Auslander-Reiten Quiver of the Category of Unstable Modules over a Sub-Hopf Algebra of the Steenrod Algebra

Abstract

For small sub-Hopf algebras of the mod-2 Steenrod algebra $\mathcal{A}_2$, we compute structure theorems for the unstable module category. As part of our structure theorems, we get a complete list of all unstable injective and projective $E(n)$ modules for $n = 0, 1$. In particular, we note that all projective objectives are injective, but the converse is not true. This gives a failure of the Faith-Walker theorem in the unstable graded case. Also, we get insight into computations of algebraic K-theory with different Waldhausen category structures.

Contents

1 Introduction

The structure of unstable modules over the mod-p Steenrod algebra, $\mathcal{A}_p$, was thoroughly exposited by Lionel Schwartz in [5], based on work by Schwartz, Lannes, Kuhn, Henn, Zarati, and others. Understanding this category helped Miller prove a version of the Sullivan fixed point conjectures. Therefore, it is of topological and algebraic interest to understand graded modules with an instability condition imposed.

By restricting to unstable B modules for B a "small" sub-hopf algebra of $\mathcal{A}_2$ we can take a hands on approach to the module theory and retain important topological information. In this paper, we will discuss the unstable $E(0)$-module category and unstable $E(1)$-module category.

First, we develop structure theorems for unstable $E(0)$-modules and unstable $E(1)$-modules. We accomplish this via Auslander-Reiten quivers. These quivers lead directly to characterizations of all projective and injective objects in these unstable module categories.

We then ask what types of Waldhausen algebraic K-theory structures we can put on the unstable module category. A Waldhausen category depends on the choice of cofibrations and weak equivalences, so for different choices of these classes of morphisms, we get different versions of Waldhausen algebraic K-theory to compute. We then compute the Grothendieck group of these categories and make note that these categories do not support a cylinder functor, an obstruction to higher degree algebraic K-theory computations.

2 Unstable $E(n)$-module theory

2.1 Notation

We follow the convention of Margolis and write $E(n) = E[e_1, e_2, ...e_n]$ for an exterior algebra on n generators with $0 < |e_1| < |e_2| < ... < |e_n|$ [2]. We want to consider the category of $E(n)$-modules that satisfy an instability condition as in [5].

Definition 2.1. We say an $E(n)$-module, M, is unstable if for all $m \in M$ $e_i(m) = 0$ for all i such that $|e_i| > |m|$.

We use mod($E(n)$), grmod($E(n)$) and $\mathcal{U}(E(n))$ to denote the categories of finitely generated $E(n)$-modules, finitely generated graded $E(n)$-modules and finitely generated unstable graded $E(n)$-modules respectively.

2.2 Auslander-Reiten Theory

A brief description of the Auslander-Reiten Theory used in this paper is needed. Auslander-Reiten theory gives a simple and elegant description of a module category by assigning a directed graph to a category where nodes represent indecomposable objects and directed edges represent irreducible morphisms. We should say in what sense the objects are indecomposable and the morphisms are irreducible.

Definition 2.2. A R-module M is indecomposable if $M \cong N \oplus L$ implies N or L is trivial.

Definition 2.3. An R-module morphism $h : M \longrightarrow N$ between indecomposable modules M and N is *irreducible if*

1. *h is not an isomorphism, and*

2. *In any commutative diagram,*

$$\begin{array}{ccc} M & \xrightarrow{h} & N \\ & \searrow^{\alpha} & \uparrow{\scriptstyle \beta} \\ & & Z \end{array} \tag{1}$$

   either α is a split monomorphism or β is a split epimorphism. In particular, $h \neq 0$ is a monomorphism or epimorphism if it is irreducible.

Recall that a split epimorphism is a epimorphism $\beta : Z \longrightarrow N$ such that there exists a $\beta' : N \longrightarrow Z$ such that $\beta \circ \beta' = \mathrm{id}_N$, and split monomorphism is dual to this.

We should also describe almost split sequences, which are important to Auslander-Reiten theory.

Definition 2.4. An R-module morphism $g : B \longrightarrow C$ is *right almost split if g is not a split epimorphism, and if $h : X \longrightarrow C$ is a R-module morphism that is also not a split epimorphism, then h factors through g. The morphism g is minimal right almost split if g is almost split and if each $t \in End(B)$ such that $ht = h$ is an isomorphism. (Minimal) left almost split morphisms are dual.*

We may now characterize almost split exact sequences.

Proposition 2.5. The following statements are equivalent for an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in mod(R).

1. f is left almost split and g is right almost split in mod(R).

2. f is minimal left almost split in mod(R).

3. g is minimal right almost split in mod(R).

Definition 2.6. An Auslander-Reiten quiver of a category, $\mathcal{C}$, is a directed graph with vertices given by indecomposable objects in $\mathcal{C}$ and edges given by irreducible morphisms in $\mathcal{C}$, where the source of the directed edge is the vertex representing the source of the morphism and the target of the directed edge is the vertex representing the target of the morphism.

For more details on Auslander-Reiten theory we recommend [1], our main resource for this material.

2.3 The structure of $\mathcal{U}(E(0))$

We will consider $E(0) \subset \mathcal{A}_2$, the mod 2 steenrod algebra and $k = \mathbb{F}_2$. We can characterize all indecomposable objects in the mod($E(0)$) since $E(0) \cong k[x]/x^2$. For any $M \in E(0)$ we have

$$M = k^{\oplus n_0} \oplus E(0)^{\oplus n_1}$$

for $n_0, n_1 \geq 0$. This is known for modules over a principal ideal domain.

If we consider the category grmod($E(n)$), the characterization remains the same except that we include suspensions; i.e. we have the same indecomposable objects and all possible n-th suspensions of those objects for $n \in \mathbb{Z}$.

From the characterization of indecomposable objects in grmod($E(0)$), we get a characterization of all indecomposable objects in $\mathcal{U}(E(0))$, by considering the allowable graded $E(0)$-modules under the instability condition.

Proposition 2.7. Any object M in the category $\mathcal{U}(E(0)$ can be written of the form

$$M = \bigoplus_i \Sigma^{m_i} k^{n_i} \oplus \bigoplus_j \Sigma^{m_j} (E(0))^{n_j} \tag{2}$$

where n_i, n_j, and $m_i \geq 0$ and $m_j \geq |e_1|$, for all i, j.

Proof. k has one generator, 1, in degree 0 and $e_1 \in E(0)$ acts trivially. We have the actions by 1 $1(1) = 1$ where $|1| = 0$ and by e_1, $e_1(1) = 0$ where $|e_1| > |1| = 0$. Thus, $\Sigma^n k$ satisfies the instability condition for all $n \geq 0$. In $E(0)$ however, $E(0)$ acts on itself by left multiplication so $e_1(1) = e_1 \neq 0$ when $|e_1| > |1|$. Thus, $E(0)$ must be suspended $|e_1|$ times before it is unstable. □

We may further describe the structure in this category via Auslander-Reiten quivers.

Proposition 2.8. The indecomposable objects in mod($E(0)$) are k and $E(0)$ and the irreducible morphisms are ϵ and η. We have a Auslander-Reiten quiver expressing this,

$$\Gamma(E(0)) : \ [E(0)] \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} [k]$$

where $\epsilon : E(0) \longrightarrow k$ sends e_1 to 0 and 1 to 1 (ϵ drops the top class) and $\eta : k \longrightarrow E(0)$ sends 1 to e_1 (η includes in the top class).

Proof. We know that all $E(0)$-modules are direct sums of k and $E(0)$ so these are the indecomposable objects in this category. We need to look at all posible maps between these objects. There are two possible ways to include k in $E(0)$ up to scalar multiplication. We wish to show only one of them is an $E(0)$-module morphism. Suppose they are both $E(0)$-module morphisms. If f is the morphism such that $f(1) = 1$, then $f(e_1 \cdot 1) = e_1 f(1) = e_1(1) = e_1 \in E(0)$; however, $f(e_1 \cdot 1) = f(0) = 0$ since e_1 acts trivially on k. Let's call the morphism which includes in the top class η, so that $\eta(1) = e_1$. Then $\eta(e_1 \cdot 1) = \eta(0) = 0$

and $\eta(e_1 \cdot 1) = \eta(e_1) = e_1^2 = 0$, so indeed this is an $E(0)$-module morphism. To show this is irreducible, we see if this map ever factors nontrivially. Suppose there exist $E(0)$-module maps $\beta : M \longrightarrow E(0)$ and $\alpha : k \longrightarrow M$, which are not isomorphisms, such that

$$\begin{array}{ccc} k & \xrightarrow{\eta} & E(0) \\ & \searrow^{\alpha} & \uparrow \beta \\ & & M \end{array}$$

commutes. Let us consider indecomposable objects only for M. If $M \cong k$, then α is an isomorphism. If $M = E(0)$ then β is an isomorphism. We then consider all $E(0)$-module morphisms from $E(0)$ to k. We could have g such that $g(e_1) = 1$ but then $g(e_1 \cdot 1) = 1$, but then E_1 acts trivially on $g(1)$ so $e_1(g(1)) = 0$. Thus, the only possibility is the map ϵ such that $\epsilon(1) = 1$. Then $\epsilon(e_1 \cdot 1) = e_1\epsilon(1) = 0 = \epsilon(e_1)$, so this is an $E(0)$ module morphism. To show ϵ is irreducible, consider a $E(0)$-module M with maps α, β such that

$$\begin{array}{ccc} E(0) & \xrightarrow{\epsilon} & k \\ & \searrow^{\alpha} & \uparrow \beta \\ & & M \end{array}$$

commutes. We assume M is an indecomposable object. If $M = E(0)$ then β is an isomorphism and if $M = k$ then α is an isomorphism. □

We proceed with a similar result about unstable $E(0)$-modules.

Proposition 2.9. The Auslander-Reiten quiver, $\Gamma(\mathcal{U}(E(0)))$, of unstable graded E(0)-modules is of the form

$$\begin{array}{rccc} \Gamma\mathcal{U}(E(0)) : & ... & \longrightarrow & [\Sigma^3 E(0)] \\ & & \swarrow & \\ & [\Sigma^3 k] & \longrightarrow & [\Sigma^2 E(0)] \\ & & \swarrow & \\ & [\Sigma^2 k & \longrightarrow & [\Sigma^1 E(0)] \\ & & \swarrow & \\ & [\Sigma^1 k] & & \\ & & & \\ & [\Sigma^0 k] & & \end{array}$$

for $n \geq 2$ where the elipses indicate that the pattern continues infinitely. We consider $e_1 \in E(0)$ to be in degree 1.

The horizontal arrows are inclusion in the top class, denoted η, and the diagonal maps drop the top class, denoted ϵ. We now expect η and ϵ to respect the grading.

Proof. We need to show that the these are all the indecomposable objects and the indecomposable morphisms. The first part follows from Proposittion 2.7. To show irreducibility of η, let M, α, β, satisfy the commutative diagram

$$\begin{array}{ccc} \Sigma^n k & \xrightarrow{\eta} & \Sigma^{n-1} E(0) \\ & \searrow^{\alpha} & \uparrow \beta \\ & & M \end{array}$$

with neither α, β an isomorphism or a trivial map. It will suffice to show for M an irreducible module. If $M = \Sigma^m E(0)$, then $m \geq n - 1$ or no $E(0)$-module map β exists. If $m \geq n$ then no $E(0)$-module map α exists. If $m = n$ then β is an isomorphism. If $M = \Sigma^m k$, then $m > n - 1$ or β is not an $E(0)$-module morphism. Also, $m \leq n$ or α is not an $E(0)$-module morphism. Thus, $m = n$ and α must be an isomorphism, which is a contradiction.

Similarly, consider M, α, β, satisfying the commutative diagram

$$\begin{array}{ccc} \Sigma^n E(0) & \xrightarrow{\epsilon} & \Sigma^n k \\ & \searrow^{\alpha} & \uparrow \beta \\ & & M \end{array}$$

where α, β are not isomorphisms or trivial. Again, we will show this result for irreducible modules M. Suppose $M = \Sigma^m E(0)$. Then $m < n$ or α is not an $E(0)$-module morphism and $m \geq n$ or β is not an $E(0)$-module morphism, so there is no m such that this diagram exists. Let $M = \Sigma^m k$. Then $m \leq n$ or α is not a $E(0)$-module morphism. Also, $m \geq n$ or else β is not an $E(0)$-module morphism. Then $m = n$, but this implies that β is an isomorphism, which is a contradiction. Thus, ϵ and η are irreducible morphisms. We may show that we have only one almost split exact sequence

$$0 \longrightarrow \Sigma^{n+1} k \xrightarrow{\eta} \Sigma^n E(0) \xrightarrow{\epsilon} \Sigma^n k \longrightarrow 0$$

for each $n \geq 1$, since η is left almost split and ϵ is right almost split. First, there is no $E(0)$-module map $\Sigma^n k \longrightarrow \Sigma^n E(0)$, so ϵ is not split. Similarly, there is no $E(0)$-module map $\Sigma^n E(0) \longrightarrow \Sigma^{n+1} k$, so η is not split. Suppose there exists an $E(0)$-module map $h : M \longrightarrow \Sigma^n k$. Again, it suffices to consider M an irreducible module. If $M = \Sigma^m E(0)$, then $m \geq n$ if h exists and the map h wil factor through $\epsilon\eta\epsilon\eta ... \epsilon$ where $\eta\epsilon$ is iterated enough times to get to the appropriate degree and if $m = n$ the map is ϵ. Since the last map in the composite is always ϵ, this will always factor through ϵ. A similar argument could be made for η. □

We notice that $\Sigma^1 k$ only has irreducible morphisms entering in the unstable setting and $\Sigma^0 k$ has no irreducible morphisms entering or exiting in the unstable setting, besides self-maps. We use this to describe the following behavior of the unstable category.

Proposition 2.10. There is a splitting of categories of $\mathcal{U}(E(0))$,

$$\mathcal{U}(E(0)) \cong \mathcal{K} \times \mathcal{D}.$$

$\mathcal{K}$ is the subcategory of $\mathcal{U}(E(0))$ with objects $M \in \mathcal{K}$ if and only if $M = \bigoplus_i k^i$ and morphisms the same as those from f.g $\mathcal{U}(E(0))$ between these objects. $\mathcal{D}$ is the subcategory of objects in f.g$\mathcal{U}(E(0))$ of the form $M = \bigoplus_i \Sigma^{m_i} k^{n_i} \oplus \bigoplus_j \Sigma^{m_j} E(0)^{n_j}$ where m_i, n_i, m_j, and $n_j \geq 1$. and the morphisms are the same morphisms between these objects in f.g$\mathcal{U}(E(0))$

Proof. Write $M \cong (\Sigma^0 k)^{\oplus a} \oplus M'$, and $N \cong (\Sigma^0 k)^{\oplus b} \oplus N'$ where M' and N' contain no summands of the form $\Sigma^0 k$. Then, we see that

$$\begin{aligned} \hom_{\mathcal{U}(E(0))} &\cong \hom_{\mathcal{U}(E(0))}((\Sigma^0 k)^{\oplus a}, (\Sigma^0 k)^{\oplus b}) \times \hom_{\mathcal{U}(E(0))}(M', N') \\ &\cong \hom_{\mathcal{K}}((\Sigma^0 k)^{\oplus a}, (\Sigma^0 k)^{\oplus b}) \times \hom_{\mathcal{D}}(M', N'). \end{aligned}$$

□

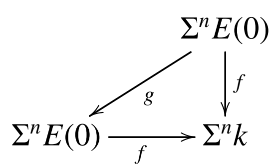
Remark. There will be a splitting of categories whenever the quiver Γ is disconnected.

This characterization is benefitial for certain computations. One immediate benefit is an understanding all of injectives and projectives in this category.

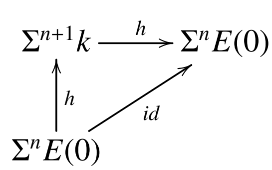
Proposition 2.11. All projectives in $\mathcal{U}(E(0))$, are of the form $\Sigma^n E(0)$ for $n \geq 1$ or $\Sigma^0 k$. All the projectives are also injective and also $\Sigma^1 k$ is injective.

Proof. The Auslander-Reiten quiver gives us all nontrivial indecomposable morphsims along with all indecomposable objects. Let us consider what the lifting property of projectives gives us in each case and find all projectives and injectives by brute force.

1. Our first claim is that $\Sigma^n E(0)$ is injective and projective. Given our one irreducible surjection $f : \Sigma^n E(0) \longrightarrow \Sigma^n k$ for $n \geq 1$, we have a commutative diagram,



   where the lifting morphism $g = Id$. The surjection $\Sigma^n k \longrightarrow \Sigma^n k$ has f as the necessary lifting, and the surjection $\Sigma^n E(0) \longrightarrow \Sigma^n E(0)$ takes the identity $\Sigma^n E(0) \longrightarrow \Sigma^n E(0)$ as a lifting. The only other case to check is the surjection $\Sigma^0 k \longrightarrow \Sigma^0 k$, but the 0 map from $\Sigma^n E(0) \longrightarrow \Sigma^0 k$, gives us the lifting we want. Note, we also can use the 0 map as a lifing in cases where we are looking for a lifting of $\Sigma^m E(0)$ where $m \neq n$ in the cases above. To show injective we follow a similar routine. The one irreducible injection we need to consider is the morphsim $h : \Sigma^{n+1} k \longrightarrow \Sigma^n E(0)$ for $n \geq 1$. We have the commutative diagram



which gives us the desired lifting. The other cases we need to conside are dual to those we did above and we simply fill in the identity, h, or the 0 morphism for both maps and we get the lifting property we need.

2. $\Sigma^0 k$ is both projective and injective. This is not hard to see because so few objects map to k. We have the diagrams,

$$\begin{array}{ccc} & & \Sigma^0 k \\ & \swarrow & \downarrow \\ \Sigma^0 k & \longrightarrow & \Sigma^0 k \end{array}$$

and

$$\begin{array}{ccc} \Sigma^0 k & \longrightarrow & \Sigma^0 k \\ \uparrow & \nearrow & \\ \Sigma^0 k & & \end{array}$$

where all the morphisms are the identity. For all other surjections and injections, the 0 morphism satisfies the lifting property; e.g.

$$\begin{array}{ccc} & & \Sigma^0 k\ . \\ & \overset{0}{\swarrow} & \downarrow{\scriptstyle 0} \\ \Sigma^n E(0) & \xrightarrow[f]{} & \Sigma^n k \end{array}$$

Since all other possibilities have the same result as this one we omit them.

3. The one special case is $\Sigma^1 k$. This object is injective, but not projective. First let us show $\Sigma^n k$ is not projective for $n \geq 1$. Consider the diagram

$$\begin{array}{ccc} & & \Sigma^n k\ . \\ & & \downarrow{\scriptstyle id} \\ \Sigma^n E(0) & \xrightarrow[f]{} & \Sigma^n k \end{array}$$

Then, there does not exist a map from $\Sigma^n k$ to $\Sigma^n E(0)$ because that would mean that f splits and this is not the case. We now show that $\Sigma^1 k$ is injective. Consider the one nontrivial injection $\Sigma^{n+1} k \longrightarrow \Sigma^n E(0)$ for $n \geq 1$, then the only map from $\Sigma^{n+1} k$ to $\Sigma^1 k$ is the 0 morphism, and this lifts by using the 0 morphism again from $\Sigma^1 k \longrightarrow \Sigma^n E(0)$ for $n \geq 1$. If we consider $\Sigma^n k \longrightarrow \Sigma^n k$, then the identity morphism gives us the desried lift for $n = 1$ and the 0 morphism works for all other $n > 1$ or for $n = 0$. If we consider $\Sigma^n E(0) \longrightarrow \Sigma^n E(0)$, then the 0 morphism gives us the desired lifting property.

4. All that remains is to show that $\Sigma^n k$ is not injective for $n > 1$. We need only consider the injection $\Sigma^n k \longrightarrow \Sigma^{n-1} E(0)$ giving us the diagram

$$\begin{array}{ccc} \Sigma^n k & \longrightarrow & \Sigma^{n-1} E(0) , \\ \uparrow{\scriptstyle id} & & \\ \Sigma^n k & & \end{array}$$

but the injection doesn't split, so there does not exist a lift. Thus, $\Sigma^n k$ not injective for $n > 1$.

□

Remark. Note that the one object keeping us from having the property that all projectives are injective and vice versa is $\Sigma^1 k$.

These properties of the category of $\mathcal{U}(E(0))$ will be necessary for putting a Waldhausen category structure on it. We proceed with similar structure theorems for $E(1)$-modules.

2.4 The structure of grmod($E(1)$)

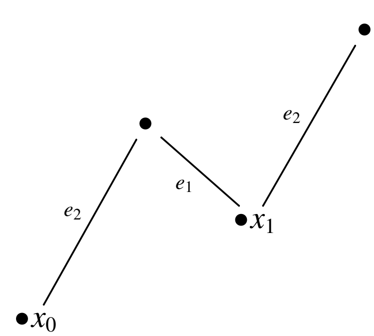
Again we will follow the convention of Margolis in [2] and let $E(n) = E[e_1, e_2, ..., e_{n+1}]$ where $E(n)$ is an exterior algebra on $n + 1$ generators and $0 < |e_1| < |e_2| < ... < |e_{n+1}|$. We wish to understand the structure of all unstable $E(1)$-modules via an Auslander-Rieten quiver. The finitely generated indecomposable objects are given by Margolis in [2] .

Proposition 2.12. Every finitely generated graded $E(1)$ module M is a summand of one of the following:

1. $\Sigma^i E(1)$, the free $E(1)$-module
2. $\Sigma^i L(0, 0, 0) = \Sigma^i k$
3. $\Sigma^i L(n, \delta_0, \delta_1)$

Proof. See Margolis [2] for a proof. □

Let us explain the notation: n indicates that there are $n + 1$ generators, δ_0 and δ_1 are either 0 or 1 describing behavior at either end. We refer to each of these as lightning flashes because of their visual description; for example L(1,0,1) is



Now we characterize all irreducible monomorphisms and epimorphisms in this category.

Proposition 2.13. Irreducible monomorphisms in grmod(E(1)) are of the form:
Non-degree shifting

1. $L(m, \delta_0, 1) \longrightarrow L(m+1, \delta_0, \delta_1)$ where $\delta_0 = 0, 1$ and $\delta_1 = 0, 1$.

Degree shifting

1. $\Sigma^{|e_2|-|e_1|} L(m, 1, \delta_1) \longrightarrow L(m+1, \delta_0, \delta_1)$ where $\delta_0 = 0, 1$ and $\delta_1 = 0, 1$
2. $\Sigma^{|e_1|} L(0, 0, 0) \longrightarrow L(0, 1, 0)$
3. $\Sigma^{|e_2|} L(0, 0, 0) \longrightarrow L(0, 0, 1)$
4. $\Sigma^{|e_1|} L(1, 0, 0) \longrightarrow E(1)$.

Proof. We will prove this by exhaustion of cases. First, let us consider the non-degree shifting monomorphisms.

1. We have four cases to consider for the first irredicucible non-degree shifting monomorphism.

   (a) For $\delta_0 = \delta_1 = 0$, we have the monomorphism $L(m, 0, 0) \longrightarrow L(m+1, 0, 0)$, which is clearly not an isomorphism. Let Z be an object in grmod($E(1)$) and α, β graded $E(1)$-module morphisms satisfying commutatitivity of the diagram 1. Suffices to consider Z an indecomposable object in grmod($E(1)$). If $Z = E(1)$, $Z = L(n, \delta_0, \delta_1)$ for $n < m$, $Z = L(n, \delta_0, \delta_1)$ for $n = m$, δ_0, δ_1 not both zero, then no such commutative diagram of $E(1)$-module morphisms exists. If $Z = L(m, 0, 0)$, or $Z = L(m+1, 0, 0)$, then the diagram commuting implies that α (resp. β) is an isomorphism and is thus a split monomorphism (resp. split epimorphism). This leaves $Z = L(n, \delta_0, \delta_1)$ where $n = m$ and δ_0, δ_1 are not both zero, or $n > m$ and δ_0, δ_1 can be zero or one. In this case, α is an inclusion and β is an epimorphism, but the inclusion $\beta' : L(m+1, 0, 0) \longrightarrow L(n, \delta_0, \delta_1)$ is a splitting. Thus, for any commutative diagram of the form 1 either α is a split monomorphism or β is a split epimorphims; i.e. the map in question is a irreducible monomorphism.

   (b) For $\delta_0 = 0, \delta_1 = 1$, we consider the monomorphism $L(m, 0, 1) \longrightarrow L(m+1, 0, 1)$, again not an isomorphism. We have a similar story for Z, α, β as before. If $Z = E(1)$, or $Z = L(n, \delta_0, \delta_1)$ for either $n < m$ or $n \geq m$, $\delta_0 = 1$, or $n = m+1$ and $\delta_0 = \delta_1 = 0$, no such commutative diagram of graded $E(1)$-modules exists. If $Z = L(m, 0, 1)$ or $Z = L(m+1, 0, 1)$ either α is a split monomorphism or β is a split epimorphism. If $Z = L(n, \delta_0, \delta_1)$ for $n > m$, $\delta_0 = 0$, then the epimorphism β is again split by the inclusion in the other direction. Thus, by exhaustion we have shown that the map is irreducible.

(c) For $\delta_0 = 1, \delta_1 = 0$, we have the same story for the monomorphism $L(m,1,1) \longrightarrow L(m,1,0)$, which is not an isomorphism. Let Z, β, α be as defined in 1 and consider only indecomposable objects Z. First, we can rule out $E(1)$ since no $E(1)$-module maps exist that make the diagram commute. Now consider the diagram

$$\begin{array}{ccc} L(m,1,1) & \longrightarrow & L(m+1,1,0)\,. \\ & \searrow^{\alpha} & \uparrow{\scriptstyle \beta} \\ & & L(n,\delta_0,\delta_1) \end{array}$$

If $n < m$ the diagram doesn't commute. When $n = m$ and $\delta_0 = 0$ or when $n = m$ and $\delta_0 = 1, \delta_1 = 0$ then the map α (resp. β) fails to be a $E(1)$-module morphism. If $n = m$, $\delta_0 = 1, \delta_1 = 1$ or if $n = m+1$, $\delta_0 = 1$, $\delta_1 = 0$, either α or β is an isomorphism. When $n > m+1$, and $\delta_0 = 1, \beta$ is a split epimorphism as before. If $n \geq m+1$, $\delta_0 = 0$, then β fails to be an $E(1)$-module morphism. Finally, if $n = m+1$, $\delta_0 = 1$, $\delta_1 = 1$ α is a split epimorphism.

(d) For $\delta_0 = 1 = \delta_1$, there is the monomorphism $L(m,1,1) \longrightarrow L(m+1,1,1)$, again not an isomorphism. Let Z, β, α be as defined in 1. Consider the diagram

$$\begin{array}{ccc} L(m,1,1) & \longrightarrow & L(m+1,1,1)\,. \\ & \searrow^{\alpha} & \uparrow{\scriptstyle \beta} \\ & & L(n,\delta_0,\delta_1) \end{array}$$

We have ruled out the case $Z = \Sigma^i E(1)$, since the commutative diagram will not exist in this case. If $n < m$, the diagram will also not commute. If $n = m$, $\delta_0 = 0$, β is not an $E(1)$-module morphisma and similarly for $\delta_1 = 0$. If $Z = L(m,1,1)$, or $Z = L(m+1,1,1)$ then α or β is an isomorphism and therefore a split epimorphism and split monomorphism. If $n > m+1$, then β is a split epimorphism. Thus, we have ruled out all cases, so the map must be an irreducible monomorphism. We note also that if $Z = \Sigma^i L(n,\delta_0,\delta_1)$ where $i > 0$, then the diagram wouldn't commute.

We now consider irreducible monomorphisms that shift degree.

1. We look at the case $\Sigma^{|e_2|-|e_1|}L(m,1,\delta_1) \longrightarrow L(m+1,\delta_0,\delta_1)$. Let δ_0, δ_1 be given. Consider the commutative diagram as in 1,

$$\begin{array}{ccc} \Sigma^{|e_2|-|e_1|}L(m,1,\delta_1) & \longrightarrow & L(m+1,\delta_0,\delta_1)\,. \\ & \searrow^{\alpha} & \uparrow{\scriptstyle \beta} \\ & & Z \end{array}$$

If $Z \cong \Sigma^i E(1)$, then the diagram doesn't commute. Let $Z = \Sigma^i L(n,\delta_0,\delta_1)$. If $n > m+1$ and $i = 0$, we see that the map α cannot be an $E(1)$-module morphism. If $n = m+1$

and $\delta_1' = \delta_1$, $\delta_0' = \delta_0$, then β is an isomorphism. If $n = m + 1$ and $i = 0$, $\delta_0' \neq \delta_0$ or $\delta_1' \neq \delta_1$, either α or β is not an $E(1)$-module morphism. If $n < m + 1$ and $i = 0$, the diagram would not commute. If $n > m$ and $i = |e_2|-|e_1|$, then α is not an $E(1)$-module morphism. If $n = m$ and $i = |e_2| - |e_1|$, $\delta_0' = 1$, $\delta_1' = \delta_1$, then α is an isomorphism. If $n = m$ and $i = |e_2| - |e_1|$, $\delta_0' = 0$, then β is not an $E(1)$-module morphism. If $n = m$ and $i = |e_2|-|e_1|$, $\delta_0' = 1$, $\delta_1' \neq \delta_1$, then either α or β is not an $E(1)$-module morphism. If $n < m$, then the diagram will not commute.

2. We now consider the case $\Sigma^i L(0, 0, 0) \longrightarrow L(0, 1, 1)$ for $i = |e_1|$ or $i = |e_2|$. Let

$$\begin{array}{ccc} \Sigma^i L(0,0,0) & \longrightarrow & L(0,1,1)\ . \\ & \searrow^{\alpha} & \uparrow \beta \\ & & Z \end{array}$$

be the commutative diagram as before. If $Z \cong \Sigma^i E(1)$, β would not be an $E(1)$-module morphism. Let $Z \cong L(n, \delta_0', \delta_1')$. If $n > 0$ then β is not an $E(1)$-module morphism. If $n = 0$, $i = |e_1|$, $\delta_0' = 1$, $\delta_1' = 0$, then β is not an $E(1)$-module morphism. If $n = 0$, $i = |e_2|$, $\delta_0' = 0$, $\delta_1' = 1$, then α is not an $E(1)$-module morphism. If $n = 0$, $\delta_0' = 0 = \delta_1'$ then either the diagram doesn't commute or if we suspend enough we get that α is an isomorphism. If $n = 0$, $\delta_0 = 1 = \delta_1'$ Then β is an isomorphism. If we suspend any other Z besides $L(0, 0, 0)$, then the diagram will not commute. Thus, we have exhausted all possiblilities and if the diagram exists then either α is a split monomorphism or β is a split epimorphism.

3. Next, we consider $\Sigma^{|e_1|} L(1, 0, 0) \longrightarrow E(1)$. Since in any given diagram as above, either α or β is not an $E(1)$-module morphism, the diagram doesn't commute, or α is a spit monomorphism or β is a split epimorphism, this map is irreducible.

□

Proposition 2.14. Irreducible epimorphisms in grmod($E(1)$) are of the form:
Non-degree shifting

1. $L(m + 1, \delta_0, \delta_1) \longrightarrow L(m + \delta_1, \delta_0, 0)$ where $\delta_0 = 0, 1$ $\delta_1 = 0, 1$.
2. $L(m + 1, 1, \delta_1) \longrightarrow L(m + 1, 0, \delta_1)$ where $\delta_1 = 1, 0$.
3. $E(1) \longrightarrow L(0, 1, 1)$

Degree shifting

1. $L(m + 1, 0, \delta_1) \longrightarrow \Sigma^{|e_2|-|e_1|} L(m, 0, \delta_1)$ where $\delta_1 = 0, 1$.

Proof. We prove an epimorphism is irreducible in the same way as in Proposition 2.13. We consider the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & {}_{\alpha}\searrow & \uparrow{\scriptstyle \beta} \\ & & Z \end{array}$$

where $X \longrightarrow Y$ is the epimorphism in question and Z is any indecomposable object such that the diagram commutes. Again, we wish to show that in each case, either α is a split monomorphism or β is a split epimorphism.

Let $X = L(m+1, \delta_0, \delta_1)$ and $Y = L(m+1, \delta_1, \delta_0, 0)$, for example. We see that if $Z \cong \Sigma^i L(n, \delta_0', \delta_1')$, for $i \neq 0$, or $n < m$, then the diagram will not commute. If $Z \cong \Sigma^i E(1)$ for any i, or if $Z \cong L(n, \delta_0', \delta_1')$ for $n > m+1$, then either α or β will not be an $E(1)$-module morphism. If $n = m+1$ or $n = m$, then either αor β is an isomorphism or α or β is split. or α or β is not an $E(1)$-module morphism. This same pattern occurs for each of the other cases.

□

We therefore have proven all we need for the following theorem.

Theorem 2.15. The Auslander-Rieten quiver of graded $E(1)$-modules, $\Gamma(E(1))$ is given in Figure 1. with irreducible monomorphisms and epimorphisms as edges and indecomposable objects as vertices.

Proof. This follows from Proposition 2.13, Proposition 2.14, and 2.12

□

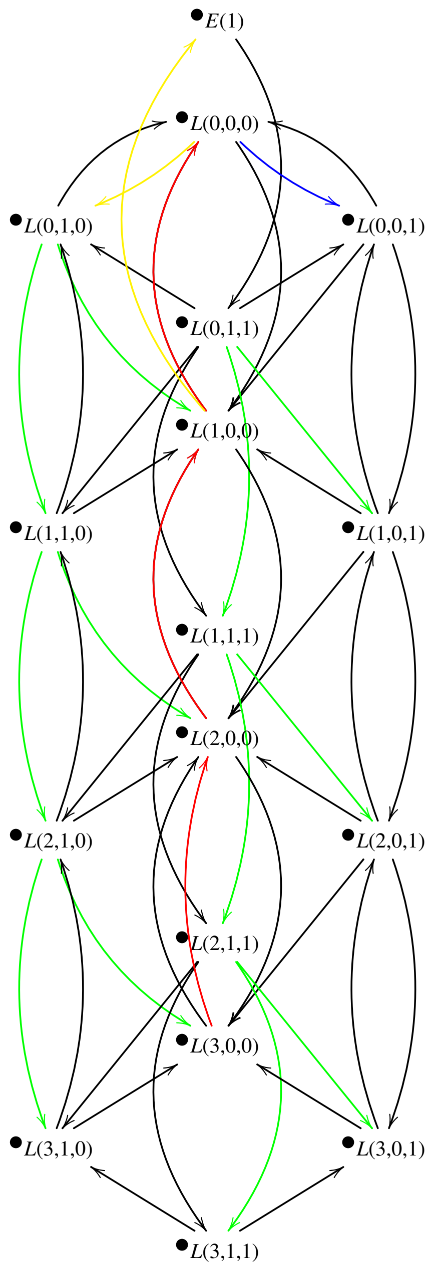


Figure 1: The Auslander-Reiten quiver of graded E(1) modules, $\Gamma(E(1))$,
Blue arrow indicates shift by degree $|e_2|$ in the source, green arrow a shift by degree $|e_2| - |e_1|$ in the source, red arrow a shift by degree $|e_2| - |e_1|$ in the target, yellow arrow a shift by degree $|e_1|$ in the source, and black arrow indicates no degree shift.

2.5 The structure of $\mathcal{U}(E(1))$

We now want to consider just the $E(1)$-modules that satisfy the instability condition given in Definition 2.1. From our classification of graded $E(1)$-modules in the previous section, we now classify modules in $\mathcal{U}(E(1))$.

Proposition 2.16. Each $M \in \mathcal{U}(E(1))$ is a summand of objects of the form:

1. $\Sigma^i E(1)$ for $i \geq 3$.
2. $\Sigma^i L(n, \delta_0, \delta_1)$ for $n \geq 1$ $\delta_0 = 0, 1$ $\delta_1 = 0, 1$ or $n = 0$ $\delta_0 = 0, 1$ $\delta_1 = 1$, $i \geq 3$
3. $\Sigma^i L(0, 0, 0)$ for $i \geq 0$
4. $\Sigma^i L(0, 1, 0)$ for $i \geq 1$.

Proof. Similar to the situation with $E(0)$-modules, we must consider which indecomposable $E(1)$ modules satisfy the instability condition. $\Sigma^i E(1)$ is not unstable for $i < |e_2|$ because $e_2(1) = e_2 \in E(1)$ so if 1 is in degree less than $|e_2|$ then the instability condition is not satisfied. Similarly, if $e_2 \in M$ then $e_2(1) = e_2$ and thus we must suspend M $|e_2|$ times in order for the instability condition to be satisfied. This takes care of $L(n, \delta_0, \delta_1)$ for $n \geq 1$ $\delta_0 = 0, 1$ $\delta_1 = 0, 1$ or $n = 0$ $\delta_0 = 0, 1$ $\delta_1 = 1$, $i \geq 3$. Since $|e_1| = 1$, if M contains e_1 then M must be suspended at least $|e_1|$ times for the same reason as above, so $L(0, 1, 0)$ must be suspended at least once to be unstable. $L(0, 0, 0) \cong k$ and since $e_i(1) = 0$ for $i = 1, 2$, it satisfies the instability condition even without suspending at all (and for any suspension of it as well). Thus, we have proven our claim. □

From our Auslander-Reiten quiver of graded $E(1)$-modules and 2.16, we can produce an Auslander-Reiten quiver of $\mathcal{U}(E(1))$. The directed graph will be the same in degrees greater or equal to $|e_2|$.

Theorem 2.17. The Auslander-Rieten quiver for the category $\mathcal{U}(E(1))$ is pictured in Figure 2. with indecomposable objects given in Proposition 2.16 and irreducible morphisms given in Proposition 2.13 and Proposition 2.14.

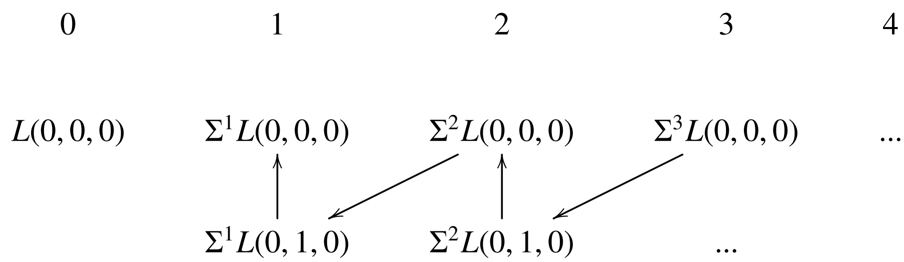


Figure 2. The figure depicts behavior of $\Gamma(\mathcal{U}(E(1)))$ in low degrees. We assume here that $|e_2| = 3$ and $|e_1| = 1$. Elipses mean that in degree $|e_2|$ the picture is the same as the Auslander Reiten quiver from Figure 1.

There are interesting projectives and injectives in this category. In $E(0)$ modules, we saw that after enough suspension, the projective (equivalently free) and injective modules are the same as in the larger graded $E(0)$-module category, but that there are also some sporadic projective and injective objects in low degrees. We see similar behavior here.

Proposition 2.18. In the category $\mathcal{U}(E(1))$, a complete list of projective and injective objects is as follows:

1. $\Sigma^n E(1)$ is projective and injective for $n \geq 3$.
2. $\Sigma^0 L(0, 0, 0) = \Sigma^0 k$ is projective and injective.
3. $\Sigma^1 L(0, 0, 0)$ is injective, but not projective.
4. $\Sigma^i L(0, 1, 0)$ is injective and projective for $i = 1, 2$

Proof. It suffices to show these results using irreducible morphisms and indecomposible objects. We show, by brute force, which objects are projective, injective or niether. First, we note that for $n \geq 3$ the quiver is exactly the same as in grmod($E(1)$), so our only projective and injective modules in degree 3 and higher are $\Sigma^i E(1)$ for $i \geq 3$, since grmod($E(1)$) is quasi-Frobenius; i.e. projectives are injective and vice versa. To see this explicitly, consider the irreducible objects that $\Sigma^i E(1)$, $i \geq 3$, maps to

$$\begin{array}{ccc} & & \Sigma^i E(1) \\ & & \downarrow \\ M & \longrightarrow & \Sigma^i L(0,1,1) \end{array} .$$

The only irreducible object in our quiver that $\Sigma^i E(1)$, $i \geq 3$, maps to is $\Sigma^i L(0, 1, 1)$. We look at which irreducible objects M surject on $\Sigma^i L(0, 1, 1)$ and see that there is only one object, $M = \Sigma^i E(1)$ which surjects onto $\Sigma^i L(0, 1, 1)$ so we can lift by the identity. Since this exausts all possible nontrivial morphisms from $\Sigma^i E(1)$ to an irreducible object, M', we see that for any other object in place of $\Sigma^i L(0,1,1)$, we would only have the zero morphism and thus the zero morphism gives a lifting such that the diagram commutes.

We now consider $\Sigma^i L(m,\delta_0,\delta_1)$ for $i \geq 3$, $m \geq 0$, $\delta_0 = 0,1$, $\delta_1 = 0,1$. The object $\Sigma^i L(m,0,\delta_1)$ is not projective for $i \geq 3$, $m \geq 1$ $\delta_1 = 0,1$ because there is the diagram

$$\begin{array}{ccc} & & \Sigma^i L(m,0,\delta_1) \\ & & \downarrow \\ \Sigma^{i+|e_1|+|e_2|} L(m-1,1,\delta_1) & \longrightarrow & \Sigma^{i+|e_1|+|e_2|} L(m-1,0,\delta_1) \end{array}$$

and no lift exists. Similarly, since both arrows in this diagram are surjections and there is no lift the other way, $\Sigma^i L(m,1,\delta_1)$ is not projective for $i \geq 3$, $m \geq 0$, $\delta_1 = 0,1$. □

The goal of classification of injective resolutions and projective resolutions leads to the following definition.

Definition 2.19. We say that a module M in $\mathcal{U}(E(1))$ has projective (resp. injective) length n if there exists a projective (resp. injective) resolution of this length. If no resolution terminates at a finite stage then we say M has length infinity.

Proposition 2.20. Every module M in $\mathcal{U}(E(1))$ has length 0 or ∞.

Proof. Injective and projective modules have length 0 rather trivially. Let P (resp. I) be an projective, then $0 \longrightarrow P \longrightarrow P \longrightarrow 0$ is a length 0 projective resolution (resp. $0 \longrightarrow I \longrightarrow I \longrightarrow 0$ is a length 0 injective resolution. Suppose M is not projective. Without loss of generality, we will assume that M is an indecomposable module. Then $M = \Sigma^3 L(n,\delta_0,\delta_1)$ for $n \geq 0$, $\delta_0 = 0,1$ $\delta_1 = 0,1$. or $M = \Sigma^1 L(0,0,0)$ since this module is injective, but not projective. For $\Sigma^1 L(0,0,0)$, we have the resolution

$$0 \longrightarrow \Sigma^1 L(0,0,0) \longrightarrow \Sigma^1 L(0,1,0) \longrightarrow \Sigma^2 L(0,1,0) \longrightarrow$$
$$\Sigma^3 E(1) \oplus \Sigma^6 E(1) \longrightarrow \Sigma^5 E(1) \oplus \Sigma^7 E(1) \oplus \Sigma^9 E(1) \longrightarrow \ldots$$

which is clearly infinite. Suppose $M = \Sigma^i L(n,\delta_0,\delta_1)$ for $i \geq 3$, $n \geq 0$. Then we have the resolution for $\delta_0 = 1$, $\delta_1 = 1$

$$0 \longrightarrow M \longrightarrow \Sigma^i E(1) \oplus \Sigma^{i+2} E(1) \oplus \ldots \oplus \Sigma^{i+2(n-1)} E(1) \longrightarrow$$
$$\Sigma^{i+3} E(1) \oplus \ldots \oplus \Sigma^{i+3+2(n-2)} E(1) \longrightarrow \ldots \longrightarrow \Sigma^{i+3(n-1)} E(1) \longrightarrow$$
$$\Sigma^{i+3(n-1)+4} E(1) \longrightarrow \Sigma^{i+3(n-1)+4+1} E(1) \oplus \Sigma^{i+3(n-1)+4+3} E(1) \longrightarrow \ldots$$

We get a similar resolution for the other cases, and thus every resolution of unstable $E(1)$-modules has length 0 or ∞.

□

3 Waldhausen category structure on $\mathcal{U}(E(0))$ and $\mathcal{U}(E(1))$

To compute K_0 of $\mathcal{U}(E(0))$, and $\mathcal{U}(E(1)$, we need to construct a Waldhausen category structure on the categories. Let us begin with a brief overview of the structure needed to define a Waldhausen category.

Definition 3.1. A *Waldhausen category,* or category with cofibrations and weak equivalences, is a pointed category, category with a 0 object, $\mathcal{C}$ with a subcategory $co\mathcal{C}$ whose morphisms are cofibrations and subcategory $w\mathcal{C}$ whose morphisms are weak equivalences satisfying the following axioms;
Cofibration axioms:

Cof1 All isomorphisms from $\mathcal{C}$ are morphisms in $co\mathcal{C}$ (This implies $co\mathcal{C}$ has all the objects of $\mathcal{C}$

Cof2 $0 \longrightarrow A$ is a cofibration for all $A \in \mathcal{C}$.

Cof3 Given a cofibration $A \longrightarrow B$ and a map $A \longrightarrow C$, then the pushout $B \cup_A C$ exists and the map $C \longrightarrow B \cup_A C$ is a cofibration. In other words the diagram,

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j} \\ C & \xrightarrow{h} & B \cup_A C, \end{array}$$

commutes with horizontal arrows in $co\mathcal{C}$.

Weak equivalence axioms:

Weq1 All isomorphisms from $\mathcal{C}$ are morphisms in $w\mathcal{C}$ ($w\mathcal{C}$ has all the objects of $\mathcal{C}$).

Weq2 Given the commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

with vertical arrows in $w\mathcal{C}$ and the arrows $(A \longrightarrow B)$ and $(A' \longrightarrow B')$ are morphisms in $co\mathcal{C}$, then the induced map $B \cup_A C \longrightarrow B' \cup_{A'} C'$ is a morphism in $w\mathcal{C}$.

For a more detailed account of Waldhausen category theory see Waldhausen's paper [6].

Theorem 3.2. The categories $\mathcal{U}(E(0))$, and $\mathcal{U}(E(1))$ admit a Waldhausen category structure with $w\mathcal{U}(E(0))$ stable equivalences and $co\mathcal{U}(E(0))$ injections. This structure, however, does not admit a cylinder functor satisfying the cylinder axiom.

As a consequence of this theorem, we know we could compute algebraic K-theory of $\mathcal{U}(E(0))$, but we do not have as many tools for computing higher algebraic K-theory groups.

Definition 3.3. We say M is stably equivalent to M' if any map ($M \longrightarrow M'$) factors through a projective.

First, we acknowledge that three of the axioms are trivially satisfied.

Lemma 3.4. The categories $\mathcal{U}(E(0))$ and $\mathcal{U}(E(1)$ satisfy cof1, cof 2, and weq 1.

Proof. Since we have an object, 0, in $\mathcal{U}(E(0))$ and $\mathcal{U}(E(1)$ and since for any $M \in \mathcal{U}(E(i))$, $i = 0, 1$, $0 \longrightarrow M$ is clearly an injection, this structure satisfies cof2. Note that isomorphisms are inclusions and also stable equivalences so weq1 and cof1 are satisfied. □

Lemma 3.5. The categories $\mathcal{U}(E(0))$ and $\mathcal{U}(E(1))$ satisfies cof3.

Proof. For cof3, note that pushouts exist in mod($E(i)$), $i = 0, 1$, and since $\mathcal{U}(E(i))$ is a subcategory of grmod($E(i)$)), we need only show that the pushout in $\mathcal{U}(E(i))$ is the same as in the larger category and that it satisfies the required property. Since the inclusion functor is exact, the inclusion will preserve pushouts. We need the pushout in the larger category to satisfy the instability condition. This is true because each element $m \in B \cup_A C$ was either in B or C, or it is 0 under the equivalence relation. Thus, for all $m \in B \cup_A C$ $xm = 0$ for $|x| > |m|$. To show the map $h : C \longrightarrow B \cup_A C$ is a cofibration, we look at the definition of the pushout. $B \cup_A C = B \coprod C/(f(a) \sim g(a))$, $a \in A$. We want to show h is injective. Throughout this argument, we use the names of the maps as in the definition. Suppose $x \in \ker(h)$, then $h(x) = [0, x] = [0, 0]$. Either x=0 or x=f(a) for some $a \in A$ and $0 = g(a)$ for that same a. But, since g is injective, $g(a) = 0$ implies $a = 0$. Thus, $f(a) = f(0) = 0$. Therefore, $x = 0$ implying h is injective. □

Lemma 3.6. The category $\mathcal{U}(E(i))$, $i = 0, 1$ satsifies weq2.

Proof. To show this, we rely on a result of Salch in [4], which says that if projective resolutions in $\mathcal{U}(E(0))$ are all of length 0 or infinity, then $\mathcal{U}(E(0))$ satisfies weq2. We see that, for $\mathcal{U}(E(0)$ all projective resolutions will be direct sums of resolutions of the following forms:

$$0 \longrightarrow P \longrightarrow P \longrightarrow 0$$

if P is projective; i.e. for $P = \Sigma^n E(0)$ $n \geq 1$ or $P = \Sigma^0 k$, with resolutions

$$0 \longrightarrow \Sigma^n k \longrightarrow \Sigma^n E(0) \longrightarrow \Sigma^{n+1} E(0) \longrightarrow ...$$

for $n \geq 1$. Where the maps are defined by composites of maps in the quiver $\Gamma(\mathcal{U}(E(0)))$.

Since these are all the irreducible $E(0)$-modules, we have shown that all projective resolutions are length 0 or ∞. We showed this same property for $\mathcal{U}(E(1))$ in Propositions 2.20

□

Proof. In order to prove our theorem, we combine the previous lemmas and our result follows. □

3.1 Computing the Grothendieck group of $\mathcal{U}(E(0))$

The Grothendieck group is the group completion of a certain monoid of objects in a category modulo some relations and it agrees with degree 0 algebraic K-theory. We describe K_0 in general before computing it for the category $\mathcal{U}(E(0))$. In general for a Waldhausen category, we have the following definition from [7].

Definition 3.7. Let $\mathcal{C}$ be a Waldhausen category, then

$$K_0(\mathcal{C})$$

is the abelian group presented with a genereator for each $C \in \mathrm{ob}\mathcal{C}$ subject to relations:

1. [C]=[C'] if there exists $(C \longrightarrow C') \in w\mathcal{C}$, and
2. [C]=[B]+[C/B] for every cofibration sequence $B \longrightarrow C \longrightarrow C/B$.

Remark. We must assume the weak equivalence classes of objects form a set.

3.1.1 Classical Grothendieck group of $\mathcal{U}(E(0))$

We recover classical $K_0(R)$ for a ring R by considering just the finitely generated projective R-modules with the Waldhausen category structure where inclusions are cofibrations and isomorphisms are weak equivalences. We write $K_0(\mathcal{U}(E(0)))$ for the classical algebraic K-theory of the category $\mathcal{U}(E(0))$.

Proposition 3.8.

$$K_0(\mathcal{U}(E(0))) = \mathbb{Z} \oplus \mathbb{Z}\{\Sigma^i | i \geq 1\}$$

Proof. To compute $K_0(\mathcal{U}(E(0)))$ we consider just finitely generated projective modules in $\mathcal{U}(E(0))$. We have $\Sigma^0 k$ and $\Sigma^i E(0)$ for $i \geq 1$. Thus,

$$K_0(\mathcal{U}(E(0))) \cong \mathbb{Z}\{k\} \oplus \mathbb{Z}\{\Sigma^i E(0) | i \geq 1\}.$$

□

3.1.2 Computing G-theory of $\mathcal{U}(E(0))$

When we consider the Waldhausen structure on a category $\mathcal{U}(E(0))$ where weak equivalences are isomorphisms, we refer to this as G-theory. Notice that we are not restricted to projective R-modules in this case. We will denote degree 0 G-theory by G_0.

Proposition 3.9.

$$G_0(\mathcal{U}(E(0)) = \mathbb{Z}[\Sigma]$$

Proof. We consider the monoid of all finitely generated modules in $\mathcal{U}(E(0))$ with direct sum under the relations given in Definition 3.7. We have one nontrivial cofibration sequence

$$0 \longrightarrow \Sigma^n k \longrightarrow \Sigma^{n-1}E(0) \longrightarrow \Sigma^{n-1}k \longrightarrow 0$$

which gives us the relation $[\Sigma^n k] + [\Sigma^{n-1}k] = [\Sigma^{n-1}E(0)]$ for $n \geq 2$. Thus we get all objects of the form $\Sigma^{n-1}E(0)$ $n \geq 2$ by adding objects of the form $\Sigma^n k$ for $n \geq 2$. This gives us that $G_0(\mathcal{U}(E(0)) \cong \mathbb{Z}[\Sigma]$ where $\Sigma^n k$ generates a copy of Z for each $n \in \mathbb{Z}_{\geq 0}$ giving us a free Z module on a countably infinite number of generators. Here $\mathbb{Z}[\Sigma]$ can be thought of as polynomial. □

Remark. Since Σ acts on the category $\mathcal{U}(E(0))$, we get an induced action of Σ on degree 0 algebraic K-theory. Thus, $G_0(\mathcal{U}(E(0))$ could be thought of as a free $\mathbb{Z}[\Sigma]$ module on one generator.

3.1.3 Computing stable G-theory of $\mathcal{U}(E(0))$

When our weak equivalences are stable equivalences, we refer to algebraic K-theory as stable algebraic G-theory denoted G_{st} as in [3]. We consider this Waldhausen category structure next.

Proposition 3.10.

$$(G_{st})_0(\mathcal{U}(E(0))) = \mathbb{Z}$$

Proof. Objects in $\mathcal{U}(E(0))$ are direct summands of $\Sigma^n k$ and $\Sigma^m E(0)$ where $n \geq 0$ and $m \geq 1$. We compute $(G_{st})_0$ by showing the relations we get on these objects. We have one nontrivial cofibration sequence

$$0 \longrightarrow \Sigma^n k \longrightarrow \Sigma^{n-1}E(0) \longrightarrow \Sigma^{n-1}k \longrightarrow 0$$

where $\Sigma^{n-1}E(0)/\Sigma^n k \cong \Sigma^{n-1}k$, and $n \geq 2$. This gives us the relation $[\Sigma^n k] + [\Sigma^{n-1}k] = [\Sigma^{n-1}E(0)]$. Note that $[\Sigma^n E(0)] = 0$ for $n \geq 1$ and $[\Sigma^0 k] = 0$, since $\Sigma^n E(0)$ for $n \geq 1$ and $\Sigma^0 k$ are projective; i.e. they are contractible under stable weak equivalence. We now have that $K_0(\mathcal{U}(E(0)))$ is generated by $[\Sigma^1 k]$, since $[\Sigma^n k] + [\Sigma^{n-1}k] = [\Sigma^{n-1}E(0)] = [0]$ for $n \geq 2$ gives the relation $[\Sigma^n k] = (-1)^{n+1}[\Sigma^1 k]$ for $n \geq 1$. Thus, we get $(G_{st})_0(\mathcal{U}(E(0)))$ is the group completion of $\mathbb{N}\{\Sigma^1 k\} \oplus \mathbb{N}\{\Sigma^2 k\}/(\Sigma^1 k \sim -\Sigma^2 k)$. We get $\mathbb{Z}\{\Sigma^1 k\} \oplus \mathbb{Z}\{\Sigma^2 k\}/(\Sigma^1 k \sim -\Sigma^2 k \cong \mathbb{Z}\{\Sigma^1 k\}$ □

Remark. Note that again we have a $\mathbb{Z}[\Sigma]$-module structure on $(G_{st})_0(\mathcal{U}(E(0)))$, where here Σ acts on $(G_{st})_0(\mathcal{U}(E(0))) \cong \mathbb{Z}$ by sending 1 to -1 .

3.1.4 The classical Grothendieck group of $\mathcal{U}(E(1)$

To calculate the classical grothendieck group of $\mathcal{U}(E(1))$, we look at the subcategory generated by the projective objects in $\mathcal{U}(E(1))$.

Proposition 3.11. $K_0(\mathcal{U}(E(1))) \cong \mathbb{Z}^3 \oplus \mathbb{Z}\{\Sigma^i | i \geq 3\}$

Proof. We have three types of projective object, $\Sigma^0 L(0,0,0)$, $\Sigma^1 L(0,1,0)$, $\Sigma^2 L(0,1,0)$, and $\Sigma^i E(1)$ for $i \geq 3$. Therefore, since $K_0(\mathcal{U}(E(1))$ is the grothendieck group completion of the monoid of these objects with direct sum as an operation, we get $K_0(\mathcal{U}(E(1)) \cong \mathbb{Z}\{k\} \oplus \mathbb{Z}\{\Sigma^i L(0,1,0) | i = 0,1\} \oplus \{\Sigma^i E(1) | i \geq 3\}$. This proves the claim. □

We see that this is not a $\mathbb{Z}[\Sigma]$ module as we saw with some of our $E(0)$-module computations.

3.1.5 Computing G-theory of $\mathcal{U}(E(1))$

To compute degree 0 G-theory, we consider the Waldhausen category with isomorphisms as weak equivalences and injections as cofibrations.

Proposition 3.12. $G_0(\mathcal{U}(E(1)) \cong \mathbb{Z}\{\Sigma^i | i \geq 0\}$

Proof. We now consider the grothendieck group generated by the monoid of all objects with operation direct sum, modulo the equivalence relation $[A] + [C] = [B]$ if there is an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. We have the almost split sequence $0 \longrightarrow \Sigma^{i+|e_1|} L(0,0,0) \longrightarrow \Sigma^i L(0,1,0) \longrightarrow \Sigma^i L(0,0,0) \longrightarrow 0$ for $i \geq |e_1|$. Thus, $[\Sigma^{i+|e_1|} L(0,0,0)] + [\Sigma^i L(0,0,0) = [\Sigma^i L(0,1,0)]$, and similarly $[\Sigma^{i+|e_2|} L(0,0,0)] + [\Sigma^i L(0,0,0)] = [\Sigma^i L(0,0,1)]$ for $i \geq |e_2|$. We then see that there are the almost split sequences

$$0 \longrightarrow \Sigma^i L(0,0,1) \longrightarrow \Sigma^i L(0,1,1) \longrightarrow \Sigma^{i+|e_1|} L(0,0,0) \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma^i L(0,1,0) \longrightarrow \Sigma^i L(0,1,1) \longrightarrow \Sigma^{i+|e_2|} L(0,0,0) \longrightarrow 0$$

So in the commutative monoid, we have $[\Sigma^{i+|e_i|} L(0,0,0)] + [\Sigma^i L(0,0,0)] + [\Sigma^{i+|e_i|} L(0,0,0)]$, $i = 1, 2$. By an inductive argument, we see that $[\Sigma^i L(n, \delta_0, \delta_1)] = \Sigma_{k,j} \Sigma_j [\Sigma^k L(0,0,0)]$ for all $i \geq 3$, $n \geq 0$, $\delta_0 = 0, 1$, $\delta_1 = 0, 1$ and for some k, j. Also, from the almost split sequence

$$0 \longrightarrow \Sigma^{i+|e_1} L(0,0,1) \longrightarrow \Sigma^i E(1) \longrightarrow \Sigma^i L(0,0,1) \longrightarrow 0$$

for $i \geq 3$, we get $\Sigma^i E(1) = \Sigma_{k,j} \Sigma_j [\Sigma^k L(0,0,0)]$ for all $i \geq 3$ and for some j, k. Thus, since all indecomposable objects can be written as a sum of $[\Sigma^i L(0,0,0)]$, we get $G_0(\mathcal{U}(E(1)) \cong \mathbb{Z}\{\Sigma^i k | i \geq 0\}$. □

3.1.6 Computing stable G-theory of $\mathcal{U}(E(1))$

Stable G-theory uses the Waldhausen cateogory where weak equivalences are stable equivalences and cofibrations are injections. This introduces a new relation to the Grothendieck group, mainly that $[A] = [0]$ if A projective.

Proposition 3.13. $(G_{st})_0(\mathcal{U}(E(1)) = \oplus_{j=3}^{\infty} \mathbb{Z}\{e^j\} / (e_j + e_{j+1} + e_{j+3} + e_{j+4})$

Proof. First, we note that we have the same split exact sequences as in the proof of Proposition 3.12. The difference in G_{st} is that we now have the relation $[P] = 0$ if P is projective since we are considering the waldhausen category with stable equivalences as weak equivalences. Thus,

$$0 \longrightarrow \Sigma^2 L(0,0,0) \longrightarrow \Sigma^1 L(0,1,0) \longrightarrow \Sigma^1 L(0,0,0) \longrightarrow 0$$

gives us that $[\Sigma^2 L(0,0,0)] + [\Sigma^1 L(0,0,0) = 0$. Also, since we have the sequence $0 \longrightarrow \Sigma^3 L(0,0,0) \longrightarrow \Sigma^2 L(0,1,0) \longrightarrow \Sigma^2 L(0,0,0) \longrightarrow 0$. $[\Sigma^3 L(0,0,0)] + [\Sigma^2 L(0,0,0)] = 0$. We know $[L(0,0,0)] = 0$. Also, $0 \longrightarrow \Sigma^{i+|E(1)|} L(0,0,1) \longrightarrow \Sigma^i E(1) \longrightarrow \Sigma^i L(0,0,1) \longrightarrow 0$ gives us that $[\Sigma^{i+|e(1)|} L(0,0,1)] + [\Sigma^i L(0,0,1)] = [\Sigma^i E(1)] = 0$ for $i \geq |e_2|$. Thus, $[\Sigma^{i+|e(1)|+|e_2|} L(0,0,0)] + [\Sigma^{i+|e_1|} L(0,0,0)] + [\Sigma^i L(0,0,0)] + [\Sigma^{i+|e_2|} L(0,0,0)] = 0$ for $i \geq 3$. We compute $(G_{st})_0(\mathcal{U}(E(1)) \cong \oplus_{j=3}^{\infty} \mathbb{Z}\{e^j\}/(e_j + e_{j+1} + e_{j+3} + e_{j+4})$ □

4 Remarks

Faith and Walker have the following well-known theorem

Theorem 4.1. In a module category, mod(R), the following are equivalent.

1. R is quasi-Frobenius
2. Every injective module is projective.
3. Every R-module embeds in a projective

In our case, the rings $E(0)$ and $E(1)$ are quasi-Frobenius, but $\mathcal{U}(E(0))$ and $\mathcal{U}(E(1))$ do not have the property "every injective module is projective." This failure has interesting consequences for computing algebraic K-theory of these categories. When computing higher algebraic K-theory groups it is useful to have extra properties in your Waldhausen category structure. For example, a cylinder functor is especially helpful to develop spectral sequence arguments. Since not every injective object is projective, we know that no cylinder functor exists by [3]. We showed here that a Waldhausen category structure does exist by considering projective resolutions of objects in the unstable graded module category, but without a cylinder functor, higher algebraic K-theory computations would be more difficult. We give here computations of classical algebraic K-theory, G-theory and stable G-theory in degree zero completing our structural picture of the categories $\mathcal{U}(E(0))$ and $\mathcal{U}(E(1))$.

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