Research Statement

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I am an algebraic topologist and my work explores the connections between algebraic topology, number theory, and geometric topology. My research program is primarily in two areas: the intersection of algebraic K-theory and chromatic homotopy theory and the intersection of equivariant topology and Hochschild homology. My work in algebraic K-theory and chromatic homotopy theory is primarily aimed towards understanding the arithmetic of ring spectra, which generalize rings. My program on equivariant topology and Hochschild homology has potential applications to geometric topology. The common thread in my research is the study of invariants of ring spectra.

One of the most highly regarded conjectures in intersection of chromatic homotopy and algebraic K-theory is the Ausoni–Rognes redshift conjecture, which generalizes the famous Lichtenbaum–Quillen conjecture. In fact, there is really an entire family of redshift conjectures and my work gives quantitative evidence for several of the redshift conjectures. The first approximation to algebraic K-theory using trace methods is Hochschild homology. I have therefore developed tools for computing Hochschild homology [AKS18].

Theorem 0.1 (Angelini-Knoll–Salch [AKS18])**.** Given a multiplicative filtration I of a commutative ring spectrum R, there is a spectral sequence whose input is the Hochschild homology of the associated graded of I which abuts to the Hochschild homology of R.

I have used this tool to give new computations of Hochschild homology (THH). Let j be the connective image of j spectrum and let taf^D denote a form of topological automorphic forms that models the second truncated Brown–Peterson spectrum $\mathrm{BP}\langle 2\rangle$

Theorem 0.2 (Angelini-Knoll [AK21], Angelini-Knoll–Culver–Höning [AKCH21])**.** I explicitly describe $\mathrm{THH}_*(j)/(v_0, v_1)$ and we explicitly describe $\mathrm{THH}_*(\mathrm{taf}^D)/(v_i, v_j)$ for $0 \leq i < j \leq 2$ where $v_0 = 3$.

Building on these computations, I have also given evidence [AKQ19, AKS20, AK22a, AK22b, AKAC+22] for the redshift conjectures. Roughly speaking the redshift conjectures state that algebraic K-theory increases chromatic complexity by one, but this depends on a notion of chromatic complexity.

One way to study chromatic complexity of a complex oriented ring spectrum is in terms of the height of the associated formal group. Though invariants such as algebraic K-theory and its approximations such as topological cyclic homology (TC), topological negative cyclic homology (TC^-) and topological periodic cyclic homology (TP) are not usually complex oriented, one observes that they become complex oriented when working over a different base. I therefore give an explicit description of the height shift in terms of the complex orientations of TC^- and TP over a certain cyclotomic base.

Theorem 0.3 (Angelini-Knoll [AK22a])**.** There is an explicit description of the complex orientation of TC^- and TP of the ring of integers in a finite extension of $\mathbb{Q}_p$ when working over a certain cyclotomic base.

Another way to study chromatic height is in terms of vanishing/non-vanishing of Morava K-theory. I have also studied redshift using this notion.

Theorem 0.4 (Angelini-Knoll–Quigley [AKQ19], Angelini-Knoll–Salch [AKS20]). For all $1 \leq \ell < m \leq n$ and $j \geq n+2$, we prove

$$K(m)_*TP(y(n)) = 0, \;\; K(\ell)_*K(y((n)) = 0, \text{and } K(j)_*K(BP\langle n\rangle) = 0.$$

This provides lower and upper bounds on the height of the ring spectra $y(n)$ and $BP\langle n\rangle$ respectively, where we say a ring spectrum R has height n if

$$n = \max\{m : K(m)_*R \neq 0\}.$$

Some of these results were later proven by complementary methods in [CMNN20, LMMT20].

The β-family is a v_2-periodic family of elements in the homotopy groups of spheres. It has been explicitly related to integral modular forms by [Lau99, Beh06, BL09]. I have therefore proven a height chromatic height analogue of results of [Ada66, Qui73], which gave the first evidence for a conjecture of Lichtenbaum [Lic73, Conjecture 2.4].

Theorem 0.5 (Angelini-Knoll [AK22b]). The mod (p, v_1)-β-family at the prime $p \geq 5$ is detected in the iterated algebraic K-theory $K(K(\mathbb{Z}))$ of the integers.

This suggests an analogue of Lichtenbaum's conjecture for iterated algebraic K-theory of rings of integers in number fields.

In joint work with Ausoni, Culver, Höning, and Rognes, we give an explicit computation of $K_*(BP\langle 2\rangle)/(p, v_1, v_2)$ proving a strong form of the redshift conjecture [Rog00] and improving on a result of [HW20] in this case.

Theorem 0.6 (Angelini-Knoll–Ausoni–Culver–Höning–Rognes [AKAC+22]). There is an explicit description of $K_*(BP\langle 2\rangle)/(p, v_1, v_2)$ as a finitely generated free $\mathbb{F}_p[v_2]$-module for all $p \geq 11$. In other words, $K(BP\langle 2\rangle)$ has pure fp-type 3.

I also have several ongoing projects extending this evidence and computational knowledge [AKHW, AKCH, AKH, AKAHR]. Work of Hahn–Raksit–Wilson [HRW22], provides a spectral sequence whose E_2-page recovers the syntomic cohomology of a ring R and often collapses in practice. In joint work with Hahn and Wilson [AKHW] we compute syntomic cohomology of truncated Brown–Peterson spectra $BP\langle n\rangle$ and connective Morava K-theory $k(n)$.

Goal 0.7. Compute syntomic cohomology of $BP\langle n\rangle$ and $k(n)$. Use this to prove the redshift conjecture and the telescope conjecture for $K(BP\langle n\rangle)$ and $K(K(n))$.

Equivariant topology has had a renaissance recently due to groundbreaking work of Hill–Hopkins–Ravenel on the Arf–Kervaire invariant problem in geometric topology. In a similar vein, my work in equivariant topology and Hochschild homology has potential applications to geometric topology. My work [AKQ21] uses equivariant structure on Hochschild homology to give a first approximation to the stable h-cobordism space of a point. The first step in this approach is to know that the Segal conjecture holds for THH of $X(n)$ so that one can compute the homotopy groups of $\mathrm{TC}(\mathbb{S})$ via descent along the map $\mathbb{S} \to X(n)$.

Theorem 0.8 (Angelini-Knoll–Quigley [AKQ21]). The Segal conjecture holds for $\mathrm{THH}(X(n))$ and consequently $\mathrm{can}\colon\; \mathrm{TC}^-(X(n)) \to \mathrm{TP}(X(n))$ is an equivalence.

My work [AKGH21] gives new perspectives on Hochschild homology (THR) and Witt vectors of rings with anti-involution.

Theorem 0.9 (Angelini-Knoll–Gerhardt–Hill [AKGH21])**.** We identify THR as the norm $N_{C_2}^{O(2)}(R)$. We prove a multiplicative double coset formula for $N_{C_2}^{O(2)}(R)$. We use this to inform a new definition of Real Witt vectors $\underline{\mathbb{W}}$ for rings with anti-involution and compute this for $\underline{\mathbb{W}}(\mathbb{Z})$ where $\mathbb{Z}$ has trivial anti-involution.

My work in progress [AKMP], gives a combinatorial approach to equivariant Hochschild homology (TH**G**). This uses the theory of crossed simplicial groups, which are a common generalization of Connes' cyclic category and simplicial groups. The input of our construction is a twisted G-ring, which generalize G-rings and rings with anti-involution.

Pre-theorem 0.1. Given a group homomorphism $\varphi\colon G \to C_2$, we prove that twisted G-rings are encoded by symmetric monoidal functors out of a (whiskered) crossed simplicial group $\Delta\Sigma \wr \varphi_+$. We use this to present a new construction of TH**G** and generalize the homology of crossed simplicial groups to spectra, which we denote $\mathrm{T}\mathbf{G}^+$ and $\mathrm{T}\mathbf{G}^-$. We then compute TH**G**, $\mathrm{T}\mathbf{G}^+$, and $\mathrm{T}\mathbf{G}^-$ of spherical group rings.

One of our goals is to study equivariant knot Floer homology using a variant of this theory.

In work in progress [AKBBK], we also use deformations to study Borel G-equivariant homotopy. In particular, we construct a category of artificial complete real motivic spectral that is the analogue of a-complete Artin–Tate real motivic homotopy theory for general groups.

Pre-theorem 0.2 (Angelini-Knoll–Behrens–Belmont–Kong [AKBBK])**.** There is a deformation

$$\mathrm{Re}\colon \mathrm{SH}^{Art} \longrightarrow \mathsf{Sp}^{BG},$$

of Borel G-equivariant homotopy theory, which is useful for computations. For example, we provide simple descriptions of computations of $RO(G)$-graded homotopy groups of kr, ko_{C_2}, and E_2, the second Lubin–Tate E-theory with genuine C_3-action.

One long-term goal of this program is to resolve the odd primary Arf–Kervaire invariant problem.

Finally, in work in progress [AKQ] we study a motivic filtration on equivariant Hochschild homology (THR) in pursuit of understanding the arithmetic of equivariant ring spectra. For rings with anti-involution, there is not very much that is known in this context. In joint work with J.D. Quigley [AKQ], we propose to use a strongly even version of the even filtration of [HRW22] to produce such a theory. We call this filtration on real topological Hochschild homology the real motivic filtration in analogy with the filtration of [HRW22] which recovers the more classical motivic filtration in certain cases.

Goal 0.10. Compute $\mathrm{THR}(BP_{\mathbb{R}}\langle 1\rangle)$ using the real motivic spectral sequence.

We also aim to understand syntomic/prismatic cohomology for suitable genuine commutative C_2-ring spectra

Goal 0.11. Develop the theory of C_2-equivariant chromatically quasisyntomic genuine commutative C_2-ring spectra and the notions of syntomic/prismatic cohomology.

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