STACKY HOMOTOPY THEORY

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1. A stack by any other name...

Largely due to the influence of Mike Hopkins and collaborators, stable homotopy theory has become closely tied to moduli theory in algebraic geometry. The benefit of this approach is that all of the tools and structure from algebraic geometry can be imported into homotopy theory. It also gives more conceptual descriptions to a lot of the structural theorems that were initially suggested by the computations of Ravenel and others. A big theme of this talk will be that there is a strong connection between Hopf algebroids and stacks and this can be developed on the level of cohomology.

First, let me recall the definition of a stack. I will be a bit handwavey, since we just saw a precise definition in the previous talk.

Definition 1.1. A prestack is a functor

$$\mathcal{M} : \mathrm{Aff}^{op} \cong \text{Comm Rings} \longrightarrow \text{Grpoids}$$

which satisfies the sheaf axiom. A stack is a prestack that satisfies a descent condition.

The sheaf condition and the descent condition depend on a notion of cover, that is a Grothendieck topology, and for this talk we will use the fpqc (faithfully flat quasi-compact) topology. For the purposes of this talk, I will use the following definition of an algebraic stack.[1]

Definition 1.2. A stack is *algebraic* if it has affine diagonal and there exists a surjective flat affine covering map $U \longrightarrow \mathcal{M}$ where U is an affine scheme. We say it is *rigidified* if it is algebraic and we have a particular choice of affine scheme that covers our stack. This choice of cover is called a presentation of the stack $\mathcal{M}$.

One way to build a stack is to have a particular Hopf algebroid (A, Γ) in mind and build the associated stack to that Hopf-algebroid. The Hopf algebroid structure ensures that the pair of corepresented functors $(\mathrm{Hom}(A, -), \mathrm{Hom}(\Gamma, -))$ takes values in groupoids (note that, in the functor-of-points approach, this is the pair $(\mathrm{Spec}\, A, \mathrm{Spec}\, \Gamma)$). This is in

[1]There is some subtlety here because the moduli stack of formal groups is not algebraic using the definition used by algebraic geometers because the map

$$\mathrm{Spec}(L) \longrightarrow \mathcal{M}_{FG}$$

is not of finite type (the ring L is infinitely generated). Aside: You can either get around this by redefining algebraic stacks to be ones where your choice of cover is faithfully flat rather than smooth (smooth implies finite type) and this is the way I will avoid the issue here. You could also define the moduli stacks associated to formal n-buds and take a colimit and that gives you that $\mathcal{M}_{FG}$ is a pro-algebraic stack (See Smithling's thesis [8] for details on this).

fact a sheaf of groupoids on the fpqc-site as long as (A, Γ) is a flat Hopf algebroid, i.e., the left unit η_L is a flat map. (See [1] for a proof of this claim.) It does not usually satisfy the descent data condition and therefore it is not usually a stack.

There is a process of "stackification," however, so that given a prestack $\mathcal{X}$ we can produce a stack $\overline{\mathcal{X}}$ associated to X and this gives an equivalence of categories

$$\mathrm{Hom}_{stacks}(\overline{\mathcal{X}}, \mathcal{Y}) \cong \mathrm{Hom}_{prestacks}(\mathcal{X}, U\mathcal{Y})$$

for any stack $\mathcal{Y}$, where U is the forgetful functor. The stackification functor is in fact a fibrant replacement functor in Hollander's model category structure on prestacks [3].

I will write $\mathcal{M}_{(A,\Gamma)}$ for the stackification of the prestack described above and refer to it as the stack associated to (A, Γ). I claim that for our purposes it is OK to restrict attention to stacks associated to Hopf algebroids. My evidence for this is that there is an equivalence of categories

$$\{\text{rigidified stacks}\} \cong \{\text{flat Hopf algebroids}\}$$

and therefore the theory of certain nice stacks is closely related to the theory of nice Hopf algebroids. (See [7] for a proof of this fact.)

2. Quasi-coherent modules on a stack

For a stack $\mathcal{M}_{(A,\Gamma)}$ associated to the Hopf algebroid (A, Γ), the idea is that comodules over the Hopf algebroid should correspond to modules (i.e. quasi-coherent sheaves) over the stack. This is Proposition 2.2. But first we have to make precise the notion of a quasi-coherent sheaves on a stack.

2.1. Preliminary definitions via the functor-of-points approach.

The definition of quasi-coherent sheaves on a stack is a straightforward generalization of the ordinary scheme version when formulated in the functor-of-points approach, so we review that first. In this approach, we identify a scheme S with its image along the Yoneda embedding, i.e. as the functor $\mathrm{Sch}^{op} \to \mathrm{Set}$ sending $T \mapsto \mathrm{Hom}_{\mathrm{Sch}}(T, S)$. It turns out that this encodes the same information as the functor $\mathrm{Rng} \to \mathrm{Set}$ sending $R \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\, R, S)$. (In particular, if $S = \mathrm{Spec}\, A$, this is the functor $R \mapsto \mathrm{Hom}_{\mathrm{Rng}}(A, R)$.)

A quasi-coherent sheaf over S (also called an S-module) corresponds to an assignment of an R-module to every element of $S(R) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\, R, S)$. For example, if $S = \mathrm{Spec}\, A$, S-modules are the same as A-modules; more precisely, given an A-module M we get a $\mathrm{Spec}\, A$-module sending an R-point $q : \mathrm{Spec}\, R \to \mathrm{Spec}\, A$ to the R-module $M \otimes_A R$. Quasi-coherent sheaves M over S are required to satisfy a slightly generalized version of this property: if $f : \mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is a morphism of schemes, and $q : \mathrm{Spec}\, R \to S$ is an R-point, then $M(\mathrm{Spec}\, R' \xrightarrow{f} \mathrm{Spec}\, R \xrightarrow{q} S) = M(\mathrm{Spec}\, R \xrightarrow{q} S) \otimes_R R'$.

We can also define pullback and pushfoward in this language. If $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a map of schemes, M is an A-module, and N is a B-module, then:

- f^*M takes an R-point $q : \mathrm{Spec}\, R \to \mathrm{Spec}\, B$ to the R-module $M(\mathrm{Spec}\, R \xrightarrow{q} \mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A) = M \otimes_A R$.

- f_*N takes an R-point $q : \operatorname{Spec} R \to \operatorname{Spec} A$ to the R-module $N \otimes_B P$ where $P = B \otimes_A R$ comes from the pullback diagram:

$$\begin{array}{ccc} \operatorname{Spec} P & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow q \\ \operatorname{Spec} B & \xrightarrow{f} & \operatorname{Spec} A \end{array} \tag{2.1}$$

Just as schemes are functors Rng $\to$ Set satisfying some properties, stacks are functors $\mathcal{M}$: Rng $\to$ Gpd satisfying some properties; quasi-coherent sheaves, pullback, and pushforward are defined essentially the same as above, in terms of R-points $\operatorname{Spec} R \to \mathcal{M}$. (Recall $\operatorname{Spec} R$ is the stack Rng $\to$ Gpd that sends a ring A to the discrete groupoid $\operatorname{Hom}(R, A)$, i.e. the groupoid where all morphisms are identities.)

Definition 2.1. A quasi-coherent sheaf over a stack $\mathcal{M}$ is an assignment of an R-module to every R-point $q : \operatorname{Spec} R \to \mathcal{M}$ satisfying the following property:

Given a map of commutative rings $R \longrightarrow R'$ and a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(R) & \xrightarrow{f} & \mathcal{M}_{(A,\Gamma)} \\ \uparrow & \nearrow_{f'} & \\ \operatorname{Spec}(R') & & \end{array}$$

there is an isomorphism

$$\alpha : \mathcal{F}(f') \cong \mathcal{F}(f) \otimes_R R'.$$

If $f : \mathcal{M} \to \mathcal{M}'$ is a morphism of stacks, N is a quasi-coherent sheaf over $\mathcal{M}$, and N' is a quasi-coherent sheaf over $\mathcal{M}'$, then:

- f^*N' takes an R-point $q : \operatorname{Spec} R \to \mathcal{M}$ to the R-module $N'(\operatorname{Spec} R \xrightarrow{q} \mathcal{M} \xrightarrow{f} \mathcal{M}')$. In particular, if $\mathcal{M} = \operatorname{Spec} A$, then $f^*N'(q) = N'(f) \otimes_A R$.
- If $\mathcal{M} = \operatorname{Spec} A$ and $\mathcal{M}'$ is algebraic, then f_*N sends an R-point $q : \operatorname{Spec} R \to \mathcal{M}'$ to the R-module $N(\operatorname{Spec} P \to \mathcal{M}) = \alpha_*\alpha'^*N = N \otimes_A P$, where $\operatorname{Spec} P$ is the pullback in

$$\begin{array}{ccc} \operatorname{Spec} P & \xrightarrow{\alpha} & \operatorname{Spec} R \\ \alpha' \downarrow & & \downarrow \\ \mathcal{M} = \operatorname{Spec} A & \xrightarrow{f} & \mathcal{M}' \end{array}$$

(The pullback is affine because $\mathcal{M}'$ is algebraic.)

Definition 2.1. A (A, Γ)-comodule is an A-module M along with an A-module map $\phi : M \longrightarrow M \otimes_A \Gamma$ which is counital and associative.

Proposition 2.2. *There is an equivalence of categories*

$$\{\textit{quasi-coherent } \mathcal{O}_{\mathcal{M}_{(A,\Gamma)}}\textit{-modules}\} \cong \{(A, \Gamma)\textit{-comodules}\}. \tag{2.2}$$

Now I will describe the functors involved in the equivalence of categories. For more details on this equivalence of categories one can read Hovey's paper [5] or Hollander's paper [2]. First, we send a quasi-coherent $\mathcal{O}_{\mathcal{M}_{(A,\Gamma)}}$-module $\mathcal{F}$ to the (A,Γ)-comodule

$$\mathcal{F}(\operatorname{Spec} A \xrightarrow{i} \mathcal{M}_{(A,\Gamma)}) \longrightarrow \mathcal{F}(\operatorname{Spec}\Gamma \xrightarrow{\eta_L} \operatorname{Spec} A \xrightarrow{i} \mathcal{M}_{(A,\Gamma)}) \cong \mathcal{F}(i) \otimes_A \Gamma$$

where i is the usual map and $\eta_L : A \longrightarrow \Gamma$ is the left unit map. The isomorphism is given by quasi-coherence. One needs to check that this map is a counital associative A-module map, but I'll leave the details as an exercise. Going back the other way, we send an (A,Γ)-comodule M to the quasi-coherent sheaf $\tilde{M}$ specified by $\tilde{M}(\operatorname{Spec} R \xrightarrow{f} \mathcal{M}_{(A,\Gamma)}) = M \otimes_A R$ and the isomorphism

$$\tilde{M}(\operatorname{Spec} R' \xrightarrow{f'} \mathcal{M}_{fg}) \cong M \otimes_A R' \cong M \otimes_A R \otimes_R R' \cong \tilde{M}(R) \otimes_R R'.$$

3. The stack of formal groups

Now it is time to introduce the main stacks of interest, the stack $\mathcal{M}_{fg}$ of formal groups and the stack $\mathcal{M}_{fg}^s$ of formal groups with strict isomorphisms. Define $\mathcal{M}_{FG}^s$ to be the quotient stack $\operatorname{Spec}(L) /\!\!/ G$ where L is the Lazard ring and G is the affine group scheme $\operatorname{Spec}(\mathbb{Z}[b_1, b_2, ...])$. The scheme G is represented by the functor that sends a commutative ring R to the power series $g \in R[[t]]$ such that $g(t) = t + b_1t^2 + b_2t^3 + ...$ This affine group scheme acts on $\operatorname{Spec}(L)$ by sending a formal group law $f \in R[[x, y]]$ and a power series g as above to $g(f(g^{-1}(x), g^{-1}(y)))$. If we keep track of the grading on L and $\mathbb{Z}[b_1, b_2, ...]$, then we also get an action by the group scheme $\mathbb{G}_m$. $\mathbb{G}_m$ and G sit inside a bigger group $G \rtimes \mathbb{G}_m =: G^+ \cong \{g \in R[[t]] | g(t) = b_0t + b_1t^2 + ...\}$ and $\mathbb{G}_m$ corresponds to the power series g such that $b_i = 0$ for $i > 0$ and G corresponds to the power series such that $b_0 = 1$. Note that a strict automorphism g of a formal group law is a power series such that $g'(0) = 1$, which is exactly one that is in G. We define $\mathcal{M}_{FG}$ to be $\operatorname{Spec}(L)/G^+$.

Now note that $\mathcal{M}_{FG}^s$ and $\mathcal{M}_{FG}$ have a natural choice of faithfully flat covering map $\operatorname{Spec}(L) \longrightarrow \mathcal{M}_{FG}^s$ and in fact one can prove that this gives a presentation of $\mathcal{M}_{FG}^s$ as a rigidified stack. It therefore corresponds to a stack associated to a Hopf algebroid and if you follow through the equivalence of categories, the stack is exactly $\mathcal{M}_{(MU_*, MU_*MU)}$.

Note that we could get $\mathcal{M}_{FG}$ as the stack associated to the Hopf algebroid

$$(L_*, L_*[b_0^{\pm}1, b_1, b_2, ...]).$$

In the rest of this section, we explore the geometry of $\mathcal{M}_{fg}$, and relate it to the (topological) chromatic story that we have seen in previous talks.

3.1. Intuition about $\mathcal{M}_{fg}$ via closed points.

One way to think of $\mathcal{M}_{fg}$ is as the stackification of the 1-truncated simplicial object

$$\operatorname{Spec} L \underset{\eta_R}{\overset{\eta_L}{\leftleftarrows}} \operatorname{Spec} W \leftleftarrows \text{degeneracies}.$$

In the functor-of-points approach, you can think of this as the simplicial object $\operatorname{Hom}_{\mathrm{Rng}}(L, -) \leftleftarrows \operatorname{Hom}_{\mathrm{Rng}}(W, -)$. If you plug in a ring R, you get a simplicial R-module which has a 0-simplex for every formal group law over R, and a 1-simplex between two formal group laws if those formal group laws are isomorphic. If $R = \mathbb{Q}$, then we've seen that there's only one isomorphism class, which means the simplicial object is connected. If $R = \overline{\mathbb{F}}_p$, we've seen that there is one isomorphism class for every height $h \in \mathbb{Z}_{>0} \cup \{\infty\}$. This means that the simplicial object has one connected component for every height. Alternatively, this means that $\mathcal{M}_{fg}$ has one closed point $\operatorname{Spec} \mathbb{Q} \to \mathcal{M}_{fg}$, and one closed point $\operatorname{Spec} \overline{\mathbb{F}}_p \to \mathcal{M}_{fg}$ for every height. The entire object $\mathcal{M}_{fg}$ reflects the way this geometry changes as one varies R.

3.2. **Height filtration.** For the rest of this section, we localize everything at a prime p and switch to working with the Hopf algebroid $(L_{(p)}, W_{(p)} = \mathbb{Z}_{(p)}[b_1, b_2, \dots])$ classifying formal groups up to strict isomorphism.

In ordinary algebraic geometry, there is a correspondence between closed subschemes and quasi-coherent ideal sheaves. In the world of stacks, there is an analogous correspondence between closed substacks and quasi-coherent ideal sheaves (over a stack). And if the stack is represented by a Hopf algebroid (A, Γ), such ideal sheaves are in correspondence with ideals of A that are (A, Γ)-comodules. Such ideals are called *invariant ideals*; equivalently, they are ideals I such that $\eta_R I \subset I\Gamma$, or $\eta_L I \subset \Gamma I$. Prime invariant ideals correspond to irreducible closed substacks.

[Irreducible] closed substacks ⇝ [Prime] qcoh. ideal sheaves ⇝ [Prime] invariant ideals

Landweber's filtration theorem classifies all the prime invariant ideals of $L_{(p)}$: they are the ideals $I_n = (p, v_1, \dots, v_n)$. The chain of inclusions $I_0 \subset I_1 \subset \dots$ gives rise to a maximal chain of closed substacks

$$\mathcal{M}_{fg}^{s} \supset \mathcal{M}_{fg}^{s,\geq 1} \supset \mathcal{M}_{fg}^{s,\geq 2} \supset \dots$$

where $\mathcal{M}_{fg}^{s,\geq n}$ is the substack cut out by the ideal I_n. There is also a chain of complementary open substacks. The following chart summarizes the properties of these substacks:

Substack		classifies:	Quotient presentation
$\mathcal{M}_{fg}^{s}$		formal groups	$\operatorname{Spec} L_{(p)} /\!\!/ \operatorname{Spec} W_{(p)}$
$\mathcal{M}_{fg}^{s,\geq n}$	closed	fg of height $\geq n$	$\operatorname{Spec} L_{(p)}/(p, \dots, v_{n-1}) /\!\!/ \operatorname{Spec} W_{(p)}$
$\mathcal{M}_{fg}^{s,\leq n} = \mathcal{M}_{fg}^{s} - \mathcal{M}_{fg}^{s,\geq n}$	open	fg of height $\leq n$	$\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm}] /\!\!/ \operatorname{Spec} W_{(p)}$
$\mathcal{M}_{fg}^{s,n} = \mathcal{M}_{fg}^{s,\geq n+1} - \mathcal{M}_{fg}^{s,\geq n}$	neither	fg of height $= n$	$\operatorname{Spec} L_{(p)}/(p, \dots, v_{n-1})[v_n^{-1}] /\!\!/ \operatorname{Spec} W_{(p)}$

All of these stacks can be defined as stackifications of a Hopf algebroid: here, $\operatorname{Spec} A /\!\!/ \operatorname{Spec} W \cong \mathcal{M}_{(A,\ A \otimes_{L_{(p)}} W_{(p)} \otimes_{L_{(p)}} A)}$.

3.3. **Relating this to the chromatic story.** If E is a cohomology theory such that E_*E is flat over the coefficient ring E_*, then (E_*, E_*E) forms a Hopf algebroid – and hence a stack

$\mathcal{M}_{(E_*,E_*E)}$. If E is also Landweber flat, then

$$\begin{aligned} E_*E &\cong MU_*E \otimes_{MU_*} E_* && \text{by LEFT} \\ &\cong E_*MU \otimes_{MU_*} E_* && E \wedge MU \cong MU \wedge E \\ &\cong MU_*MU \otimes_{MU_*} E_* \otimes_{MU_*} E_* && \text{by LEFT} \end{aligned}$$

(The consequence of switching the factors E and MU in the second isomorphism is that, in the last line, MU_* acts on the second factor of MU_*MU, instead of the first. That is, the action is via η_R instead of η_L.)

So in this situation, $(E_*, E_*E) = (E_*, E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_*)$, which means that there is a natural map $(MU_*, MU_*MU) \to (E_*, E_*E)$. This gives rise to a map of stacks $\mathcal{M}_{(E_*,E_*E)} \to \mathcal{M}_{(MU_*,MU_*MU)} = \mathcal{M}^s_{fg}$. An alternative statement of the Landweber exact functor theorem given at the end of this lecture implies that this map is flat.

For example, let $E = E(n)$ be Johnson-Wilson E-theory. Recall that this is Landweber exact, and that $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$. Looking at the chart of substacks of $\mathcal{M}^s_{fg}$, we see that this corresponds to the open substack $\mathcal{M}^{s,\leq n}_{fg}$, and the natural map to $\mathcal{M}^s_{fg}$ is just the inclusion.

This can be regarded as an algebraic manifestation of *chromatic tower*:

$$X \to \cdots \to L_{E(n)}X \to L_{E(n-1)}X \to \ldots .$$

The idea is that, in our language of stacks over $\mathcal{M}^s_{fg}$, $L_{E(n)}$ is supposed to correspond to "restriction to the $\mathcal{M}^{s,\leq n}_{fg}$ part". Going along this intuition, if $i_n : \mathcal{M}^{s,\leq n}_{fg} \hookrightarrow \mathcal{M}^s_{fg}$ denotes the inclusion and M is a quasi-coherent sheaf over $\mathcal{M}^s_{fg}$ (maybe MU_*X), we might try to define a filtration

$$M \to \cdots \to M_n \to M_{n-1} \to \cdots \to M_0$$

where $M_n = (i_n)_*(i_n)^*M$ (that is, you restrict to $\mathcal{M}^{s,\leq n}_{fg}$ and then use the inclusion to get a sheaf over the right stack). This ends up not quite being the right filtration – actually, we should be working in the derived category instead, and use $R(i_n)_*$ instead of $(i_n)_*$ (we don't need to do this to $(i_n)^*$, since that functor is already exact). This is called the *algebraic chromatic filtration*, and there is a convergence theorem that says that $M \to \operatorname{holim} R(i_n)_*(i_n)^*M$ is an equivalence if M is coherent. (This finiteness condition mirrors the topological case, which says that the chromatic filtration converges when X is a finite p-local spectrum.) More details can be found in [7] and [1, §5, §8].

4. A stacky doomsday device

Hopf algebroids are interesting to topologists because they arise naturally as the pair (E_*, E_*E) where E is your favorite (flat commutative) ring spectrum. We love these Hopf algebroids because their cohomology is the input for the Adams-Novikov spectral sequence

$$E_2^{*,*} = \operatorname{Ext}^{*,*}_{(E_*,E_*E)}(E_*, E_*X) \Rightarrow \pi_*(\hat{X}_E).$$

I will mainly care about the case $E = MU$. Now I want to prove the following claim.

Proposition 4.1.

$$E_2^{p,*} = \operatorname{Ext}^{p,*}_{(MU_*,MU_*MU)}(MU_*, MU_*X) \cong H^p(\mathcal{M}^s_{FG}; \mathcal{F}_X)$$

where $\mathcal{F}_X$ is a quasi-coherent sheaf associated to X, and $\mathcal{M}_{FG}^s$ is the moduli stack of formal groups with strict isomorphisms.

With the equivalence of categories of Equation 2.2, we can specify the quasi-coherent sheaf associated to a spectrum (or space) X. We simply consider the (MU_*, MU_*MU)-comodule MU_*X and consider its image under the equivalence of categories. We denote this quasi-coherent sheaf $\mathcal{F}_X$. Note that since we are considering an ungraded (MU_*, MU_*MU)-comodule we are forgetting the grading on MU_*X. There is a way to get around this by using tensor powers of a locally free of rank one sheaf ω. This sheaf is specified by the map

$$\mathcal{M}_{FG} \longrightarrow B\mathbb{G}_m$$

since maps into $B\mathbb{G}_m$ classify line bundles on a stack.

Now we need to show how $H^*(\mathcal{M}_{FG}^s; \mathcal{F}_X)$ computes the input of the Adams-Novikov spectral sequence. When we compute stack cohomology it is much easier to work with the Čech-resolution of the stack. Given a rigidified stack, we can form the Čech resolution associated to the specified cover $\text{Spec}(A) \longrightarrow \mathcal{M}$. In this case $A = MU_*$ and $\mathcal{M} = \mathcal{M}_{(MU_*,MU_*MU)}$. The Čech resolution is the simplicial scheme

$$\mathcal{M} \longleftarrow \text{Spec}(A) \leftleftarrows \text{Spec}(A) \times_{\mathcal{M}} \text{Spec}(A) \Lleftarrow \text{Spec}(A) \times_{\mathcal{M}} \text{Spec}(A) \times_{\mathcal{M}} \text{Spec}(A) \Lleftarrow \ldots$$

For any $\mathcal{O}_{\mathcal{M}}$-module sheaf we produce a spectral sequence,

$$H^s(\text{Spec}(A)^{\times_{\mathcal{M}} t}; \mathcal{F}) \Rightarrow H^{s+t}(\mathcal{M}; \mathcal{F})$$

where the left hand side is cohomology of an affine scheme and the right hand side the derived functors of global sections of a $\mathcal{O}_{\mathcal{M}}$-module sheaf. When $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{M}}$-module Serre's theorem says that $H^s(\text{Spec}(A)^{\times_{\mathcal{M}} t}; \mathcal{F}) \cong 0$ for $s > 0$ so the spectral sequence is only nonzero on the zero line and it collapses with input $H^0(\text{Spec}(A)^{\times_{\mathcal{M}} t}; \mathcal{F})$. This is just the homology of the complex we get by evaluating $\mathcal{F}$ on the Čech resolution and taking the alternating sign chain complex

$$\mathcal{F}(\text{Spec}(A)) \longrightarrow \mathcal{F}(\text{Spec}(A) \times_{\mathcal{M}} \text{Spec}(A)) \longrightarrow \mathcal{F}(\text{Spec}(A) \times_{\mathcal{M}} \text{Spec}(A) \times_{\mathcal{M}} \text{Spec}(A)) \longrightarrow \ldots.$$

In the case where $\mathcal{F} = \mathcal{F}_X$ and $(A, \Gamma) = (MU_*, MU_*MU)$, this gives

$$MU_*X \longrightarrow MU_*MU \otimes_{MU_*} MU_*X \longrightarrow MU_*MU \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*X \longrightarrow \ldots$$

since $\mathcal{F}(\text{Spec}(MU_*) \otimes_{\mathcal{M}_{(MU_*,MU_*MU)}} \text{Spec}(MU_*) \cong \mathcal{F}(\text{Spec}(MU_*MU)) \cong MU_*MU \otimes_{MU_*} MU_*X$. This follows by the quasi-coherence of $\mathcal{F}_X$, the affineness of the covering map

$$\text{Spec}(MU_*) \longrightarrow \mathcal{M}_{(MU_*,MU_*MU)},$$

which assures us that the pullback is affine, and the flatness of MU_*MU over MU_*.

Now, we see that

$$H^0(\text{Spec}(MU_*)^{\times_{\mathcal{M}_{(MU_*,MU_*MU)}} k}; \mathcal{F}_X) \cong H^k(C_\bullet) \cong \text{Ext}^{k,*}_{(MU_*,MU_*MU)}(MU_*, MU_*X).$$

The only problem with this is that we can't pick out the internal graded pieces of $E_2^{*,*}$. To get at those pieces, we need to use the $\mathbb{G}_m$-action that comes from the fact that these objects also have a grading. If we take $\mathbb{G}_m$-fixed points of $H^*(\mathcal{M}_{FG}^s; \mathcal{F}_X)$ then we get the internal grading degree-zero pieces. We then just need to take coefficients in the tensor powers of an invertible sheaf ω tensored with our sheaf $\mathcal{F}_X$

$$H^p(\mathcal{M}^s_{FG}; \mathcal{F}_X \otimes \omega^{\otimes q}) \cong \mathrm{Ext}^{p,q}_{(MU_*, MU_*MU)}(MU_*, MU_*X).$$

Application 4.1. In section 3.3, we saw that the open inclusion of substacks $\mathcal{M}^{s,\leq n}_{fg} \hookrightarrow \mathcal{M}_{fg}$ corresponded to the Bousfield localization $L_{E(n)}X$ of a p-local space X. This gives rise to a map of spectral sequences

$$\begin{array}{ccc} H^*(\mathcal{M}^s_{fg}, \mathcal{F}_X) & \Longrightarrow & \pi_* X_{(p)} \\ \downarrow & & \downarrow \\ H^*(\mathcal{M}^{s,\leq n}_{fg}, \mathcal{F}_X) & \Longrightarrow & \pi_* L_{E(n)} X. \end{array}$$

5. Landweber exactness, the stacky way

Landweber flatness is a criterion on an L-module M guaranteeing that $X \mapsto MU_*(X) \otimes_L M$ is a homology theory. This criterion is weaker than requiring M to be flat over L. The slogan for this final section is that M is Landweber flat if it is flat “in the stacky sense”.

5.1. **Flatness.** We will eventually show that Landweber flatness has something to do with flatness of quasi-coherent sheaves over $\mathcal{M}_{fg}$, so first we need to discuss flatness in this context.

Proposition 5.1. *A quasi-coherent sheaf M on a stack $\mathcal{M}$ is flat (i.e. $- \otimes_{\mathcal{O}_{\mathcal{M}_{fg}}} M$ is exact) iff, for every R-point $q : \mathrm{Spec}\, R \to \mathcal{M}$, the R-module $M(q) = q^*M$ is flat over R.*

Proof. Flatness is a local condition. The condition about R-points is essentially checking flatness locally. □

Lemma 5.2. *Suppose M is an L-module. Then u_*M is flat over $\mathcal{M}_{fg}$ iff $u^*u_*M = M \otimes_L W$ is flat over L.*

(Note: being careful with Definition 2.1, we see that in $M \otimes_L W$, L acts on W via the right unit η_R.)

Proof. One direction is obvious: if u_*M is flat over $\mathcal{M}_{fg}$, then the L-point $u^*u_*M = M \otimes_L W$ is flat over L.

Conversely, we have to check that, for every R-point $\mathrm{Spec}\, R \xrightarrow{q} \mathcal{M}_{fg}$, q^*u_*M is flat over L; we only know this for the L-point u. Since flatness is a local condition, we may assume that the R-point corresponds to a coordinatizable formal group law, and so q factors as $\mathrm{Spec}\, R \xrightarrow{f} \mathrm{Spec}\, L \xrightarrow{u} \mathcal{M}_{fg}$ (so f corresponds to the map $L \to R$ classifying the coordinatizable formal group law). Then we have to show that $q^*u_*M = f^*u^*u_*M = f^*(M \otimes_L W) = (M \otimes_L W) \otimes_L R$ is flat over R. It can be seen easily by writing out exact sequences that this follows from flatness of $M \otimes_L W$ over L. □

5.2. **Landweber exact functor theorem, take 2.** Now we're ready to talk about the Landweber exact functor theorem. Let's recall the setup: given an L-module M, we are wondering when the functor $X \mapsto MU_*X \otimes_L M$ is a homology theory. The only one of the Eilenberg-Steenrod axioms that causes an impediment is the LES-of-a-pair axiom: given an excisive pair, there is a long exact sequence in $MU_*(-)$, but a priori this only stays exact in $MU_*(-) \otimes_L M$ "homology" if M is flat over L. But requiring such flatness is too restrictive for our purposes; the Landweber exact functor theorem uses the fact that $-\otimes_L M$ only needs to preserve exactness of certain sequences of L-modules – namely, those of the form MU_*X – to prove a less restrictive, and more useful, criterion on M.

The key observation is that L-modules of the form MU_*X are comodules, and hence they come from quasi-coherent sheaves over $\mathcal{M}_{fg}$. More precisely, $MU_*X = u^*\mathcal{F}_X$ where u is the canonical map $u : \operatorname{Spec} L \to \mathcal{M}_{fg}$ and $\mathcal{F}_X$ is the $\mathcal{M}_{fg}$-module associated to the space X. So we don't have to prove that $- \otimes_L M$ preserves exactness for all exact sequences of L-modules, but only those of the form u^*(exact sequence of $\mathcal{O}_{\mathcal{M}_{fg}}$-modules) where $u : \operatorname{Spec} L \to \mathcal{M}_{fg}$ is as above.

The classical Landweber flatness criterion is:

Criterion 5.3. The sequence $(p, v_1, v_2, \dots)$ is a regular sequence on M; that is, multiplication by v_i is injective on $M/(p, v_1, \dots, v_{i-1})$.

But I claim that this can be formulated in stacky language as follows:

Criterion 5.4. Let $u : \operatorname{Spec} L \to \mathcal{M}_{fg}$ be the canonical map. Then u_*M is flat as an sheaf over $\mathcal{M}_{fg}$.

Theorem 5.5. *Criterion 5.4 $\implies$ $MU_*(-) \otimes_L M$ is a homology theory.*

The following proof is essentially [6, Lecture 15, Proposition 5].

Proof. Since $\mathcal{M}_{fg} = \mathcal{M}_{(L,W)}$, we have a pullback diagram:

$$\begin{array}{ccc} \operatorname{Spec} W & \xrightarrow{\eta_R} & \operatorname{Spec} L \\ {\scriptstyle \eta_L}\downarrow & & \downarrow{\scriptstyle u} \\ \operatorname{Spec} L & \xrightarrow{u} & \mathcal{M}_{fg} \end{array} \tag{5.1}$$

Suppose $MU_*X_1 \to MU_*X_2 \to MU_*X_3$ is an exact sequence (e.g. if $X_1 \to X_2 \to X_3$ is a cofiber sequence). We want to show that applying $- \otimes_L M$ to this is still exact. As mentioned above, this sequence can be written $u^*\mathcal{F}_{X_1} \to u^*\mathcal{F}_{X_2} \to u^*\mathcal{F}_{X_3}$. Since η_L is flat (W is a polynomial algebra over L),

$$\cdots \to u^*\mathcal{F}_{X_i} \otimes_L M \to \dots \text{ is exact } \iff \eta_L^*(\cdots \to u^*\mathcal{F}_{X_i} \otimes_L M \to \dots) \text{ is exact.} \tag{5.2}$$

We have:

$$\begin{aligned}
\eta_L^*(u^*\mathcal{F}_{X_i} \otimes_L M) &\cong \eta_L^* u^*\mathcal{F}_{X_i} \otimes_W \eta_L^* M \\
&\cong \eta_R^* u^*\mathcal{F}_{X_i} \otimes_W \eta_L^* M && \text{by (5.1)} \\
&\cong (u^*\mathcal{F}_{X_i} \otimes_L^{\eta_R} W) \otimes_W (M \otimes_L^{\eta_L} W) \\
&\cong u^*\mathcal{F}_{X_i} \otimes_L^{\eta_R} (M \otimes_L^{\eta_L} W)
\end{aligned}$$

where this last $\otimes_L^{\eta_R}$ means that $\ell \in L$ acts on $(M \otimes_L^{\eta_L} W)$ via $m \otimes w \mapsto m \otimes \eta_R(\ell)w$. Therefore, this is $u^*\mathcal{F}_{X_i} \otimes_L^{\eta_L} (M \otimes_L^{\eta_R} W)$, and

$$(5.2) \iff \cdots \to u^*\mathcal{F}_{X_i} \otimes_L (M \otimes_L^{\eta_R} W) \to \dots \text{ is exact.}$$

This sequence is exact by Lemma 5.2. □

Theorem 5.6 (Landweber exact functor theorem)**.** *Criterion 5.4* $\iff$ *Criterion 5.3.*

Proofs of the Landweber exact functor theorem in the language of stacks can be found in [6, Lecture 16] and [4, §21]. We give a brief sketch of the argument.

Proof sketch. The goal is to show that $\mathrm{Tor}_i(u_*M, \mathcal{N}) = 0$ for all quasi-coherent sheaves $\mathcal{N}$ over $\mathcal{M}_{fg}$ and all $i > 0$. The main idea is to reduce to proving a statement about quasi-coherent sheaves over $\mathcal{M}_{fg}^n$, because *all* such sheaves are flat.

Step 1: Show that that there is a faithfully flat cover $\mathrm{Spec}\,\mathbb{F}_p \to \mathcal{M}_{fg}^n$. This implies that all quasi-coherent sheaves over $\mathcal{M}_{fg}^n$ are flat, because flatness over $\mathcal{M}_{fg}^n$ is equivalent to flatness of the pullback along a faithfully flat cover, and all $\mathbb{F}_p$-modules are flat over $\mathbb{F}_p$.

Step 2: Use the long exact sequences associated to short exact sequences $0 \to u_*M \xrightarrow{p} u_*M \to u_*M/p \to 0$, etc., to reduce to proving $\mathrm{Tor}_{k+1}(v_k^{-1}(u_*M)/(p, \dots, v_{k-1}), \mathcal{N}) = 0$.

Step 3: Show that this Tor is isomorphic to a Tor term in the category of $\mathcal{O}_{\mathcal{M}_{fg}^n}$-modules, so we are done by Step 1. The idea is that the "forgetful" functor $f_* : \mathcal{O}_{\mathcal{M}_{fg}^n} \to \mathcal{O}_{\mathcal{M}_{fg}}$ has a left adjoint $f^* : \mathcal{O}_{\mathcal{M}_{fg}} \to \mathcal{O}_{\mathcal{M}_{fg}^n}$ taking $N \mapsto v_n^{-1}N/(p, \dots, v_{n-1})$. This can be used to show the following explicitly:

$$\begin{aligned}
\mathrm{Tor}_{k+1}^{\mathcal{O}_{\mathcal{M}_{fg}}}(v_k^{-1}(u_*M)/(p, \dots, v_{k-1}), \mathcal{N}) &= \mathrm{Tor}_{k+1}^{\mathcal{O}_{\mathcal{M}_{fg}}}(f_*f^*(u_*M), \mathcal{N}) \\
&\cong \mathrm{Tor}_{k+1}^{\mathcal{O}_{\mathcal{M}_{fg}}}(f_*f^*(u_*M), f_*f^*\mathcal{N}) \\
&\cong \mathrm{Tor}_{k+1}^{\mathcal{O}_{\mathcal{M}_{fg}^n}}(f^*(u_*M), f^*\mathcal{N})
\end{aligned}$$

where the last isomorphism uses the fact that f_* is exact. □

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