Research Statement

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I am an algebraic topologist and my work explores the connections between algebraic topology, number theory, geometric topology. The tools I use to investigate these connections are invariants such as algebraic K-theory and topological Hochschild homology. My work is rooted in the chromatic approach to homotopy theory and the equivariant approach to studying geometric topology.

Algebraic K-theory assigns abelian groups $\mathrm{K}_s(R)$ to a ring R. In low degrees, it recovers invariants like the class group, the Picard group, and the Brauer group. In high degrees, algebraic K-theory groups encode special values of Dedekind zeta functions, exhibiting deep connections to number theory. The low degree algebraic K-theory groups also recover obstruction groups in geometric topology like Wall's finiteness obstruction. The higher algebraic K-theory groups encode the homotopy groups of the stable h-cobordism space of a manifold. Topological Hochschild homology (THH) is a first approximation to algebraic K-theory. It also has important connections to number theory through its connection to the de Rham–Witt complex and integral p-adic Hodge theory. Additionally, THH can be used to construct knot Floer homology theories and shed light on geometric topology.

In chromatic homotopy theory, we study homotopy groups by filtering them into families of v_n-periodic elements. Larger n corresponds to periodicity of longer wavelength, higher chromatic complexity, and deeper arithmetic information. The Lichtenbaum–Quillen conjectures posed in the 1970's exhibit deep connections between number theory, algebraic K-theory, and v_1-periodicity. Ausoni–Rognes extended these conjectures to higher chromatic complexities in the 2000's. The imprecise version of the Ausoni–Rognes conjectures states that algebraic K-theory increases chromatic complexity by one and therefore they are referred to as the red-shift conjectures. These conjectures are under active investigation with a number of important recent advances in the last five years.

My research gives evidence for the red-shift conjectures from several different perspectives. I have done this by developing tools for computing THH [AK18], giving new computations of THH using these tools [AK17, AK21b, AKCH21b], and building on these computations to give evidence [AK18, AKS20, AKQ19, AK21a] for the red-shift conjectures in algebraic K-theory. I also have several ongoing projects extending this evidence and computational knowledge [AKAC$^+$21, AKHW21, AKCH21a]. These results are summarized in Section 1.

Equivariant topology has had a renaissance recently due to groundbreaking work of Hill, Hopkins, and Ravenel on the Arf–Kervaire invariant one problem in geometric topology. My work uses THH and genuine equivariant homotopy theory to approach questions in geometric topology. My work [AKQ21] uses equivariant homotopy and THH to give a first step towards computing the stable h-cobordism space of a point. My work in progress [AKGH21, AKMP21] gives new perspectives on equivariant THH and equivariant Witt vectors. One goal of this program is to provide new constructions of equivariant knot Floer homology. In work in progress [AKBBK21], we also aim to use a synthetic analogue of motivic homotopy theory to study C_3-equivariant homotopy, with the goal of resolving the 3-primary Arf–Kervaire invariant problem. These results are summarized in Section 2.

1 Algebraic K-theory and applications to number theory

This section is split into four subsections: Section 1.1 a brief overview of the red-shift conjectures, Section 1.2 a brief overview of trace methods, Section 1.3 a summary of results, and Section 1.4 a summary of work in progress.

1.1 Higher Lichtenbaum–Quillen conjectures

Broadly, the Ausoni–Rognes red-shift conjectures are about the arithmetic of ring spectra. Classically, work of Adams [Ada66] and Quillen [Qui72] showed a v_1-periodic family known as the α-family is detected in $\mathrm{K}_{4k-1}(\mathbb{Z})$. This was the first evidence for a conjecture of Lichtenbaum that

$$|\mathrm{K}_{4k-2}(\mathcal{O}_F)|/|\mathrm{K}_{4k-1}(\mathcal{O}_F)| = \zeta_F(-2k+1)$$

up to a sign and power of two, where $\mathcal{O}_F$ is the ring of integers in a finite extension of $\mathbb{Q}$ and ζ_F is a the associated Dedekind zeta function. My work [AK18] gives the first indication of a higher chromatic height analogue of this conjecture.

Question 1.1. [Higher Lichtenbaum] Is there a precise relation between $\pi_*K(K(\mathcal{O}_F))$ and modular forms over $\mathcal{O}_F$?

The Ausoni–Rognes red-shift conjectures generalize the Lichtenbaum–Quillen conjectures to higher chromatic heights and they remain under active investigation with some significant advances in the last five years. In the phrasing of Waldhausen [Wal84], the Lichtenbaum–Qullen conjectures state that the map

$$\pi_s \mathrm{K}(R[1/p])_{(p)} \longrightarrow \pi_s L_1^f \mathrm{K}(R[1/p])_{(p)}$$

is an isomorphism for sufficiently large s. In this case we say that the localization map $\mathrm{K}(R[1/p])_{(p)} \longrightarrow L_1^f \mathrm{K}(R[1/p])_{(p)}$ is truncated. One phrasing of the Ausoni–Rognes red-shift philosophy is the following question.

Question 1.2. [Higher Lichtenbaum–Quillen] Let R be a ring spectrum. If the localization map $R_{(p)} \longrightarrow L_n^f R_{(p)}$ is truncated, then is the localization map $\mathrm{K}(R)_{(p)} \longrightarrow L_{n+1}^f \mathrm{K}(R)_{(p)}$ truncated?

This reduces to the Lichtenbaum–Quillen conjecture when $n = 0$. A weak form of this conjecture has been posed for $\mathbb{E}_2$ ring spectra and it is a step towards Question 1.2.

Question 1.3. [Red-shift philosphy] Given an $\mathbb{E}_2$ ring spectrum R, then $L_{K(n)}R = 0$ if and only if $L_{K(n+1)}K(R) = 0$.

One of the original phrasings of the Ausoni–Rognes red-shift conjectures appeared in [Rog00]. This used the notion of pure FP-type based on work of [MR99]. There are finite complexes $F(n)$ for each n and prime p that are equipped with a v_n self-map

$$v\colon\ \Sigma^d F(n-1) \longrightarrow F(n-1).$$

We say that X has pure FP-type n if $F(n-1)_*X$ is a finitely generated free $P(v)$-module.

Question 1.4. If R has pure FP-type n, then does $K(R)$ has pure FP-type $n+1$?

In my work and my work in progress I give evidence for each of these questions, summarized in Sections 1.3–1.4.

1.2 Trace methods

My approach to algebraic K-theory of a connective ring spectrum R uses the theory of trace methods developed by [BHM93], where one computes algebraic K-theory by successive approximations. The first approximation to algebraic K-theory is topological Hochschild homology. Classically, one uses genuine equivariant homotopy theory and genuine cyclotomic structure to construct $\mathrm{TR}(R)$ from topological hochschild homology and then define p-complete TC as the homotopy equalizer

$$\mathrm{TC}(R)^\wedge_p = \mathrm{eq}\left(\mathrm{TR}(R) \underset{F}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{TR}(R) \right).$$

The groups $\mathrm{TR}_*(R)$ are interesting in their own right because they recover the de Rham Witt complex of a commutative ring [HM03].

Groundbreaking work of Nikolaus–Scholze [NS18] simplified the computational approach to topological cyclic homology $\mathrm{TC}(R)$ by describing TC as the homotopy equalizer

$$\mathrm{TC}(R) = \mathrm{eq}\left(\mathrm{TC}^-(R) \underset{\varphi}{\overset{can}{\rightrightarrows}} \mathrm{TP}(R)^\wedge \right)$$

of two maps where $(-)^\wedge$ denotes pro-finite completion. The topological negative cyclic homology $\mathrm{TC}^-_*(R)$ and topological periodic cyclic homology $\mathrm{TP}_*(R)$ are topological analogues of classical algebraic negative cyclic homology and periodic cyclic homology and they are easier to calculate because they do not rely on genuine equivariant homotopy theory. The intermediate approximations $\mathrm{TC}^-(R)$ are $\mathrm{TP}(R)$ to algebraic K-theory theory, also have deep relations to number theory through work of Bhatt–Morrow–Scholze [BMS19] on integral p-adic Hodge theory.

Finally, Dundas–Goodwillie–McCarthy [DGM13] proved that topological cyclic homology is a close approximation to algebraic K-theory in the sense that if $f\colon A \to B$ is a map of ring spectra such that $\pi_0 f$ is surjective with nilpotent kernel and $E(A, B)$ is the fiber of $E(f)$ for $E \in \{K, \mathrm{TC}\}$, we have an equivalence $\mathrm{K}(A, B) \simeq \mathrm{TC}(A, B)$. This gives an explicit computational approach to algebraic K-theory, which has led to numerous advances in the subject, for example [HM03, AR02, HW20].

1.3 Results on higher Lichtenbaum–Quillen conjectures

One of my main goals has been to give evidence for the Ausoni–Rognes red-shift philosophy. I am motivated by understanding the arithmetic of ring spectra, which generalize rings.

There are v_n-periodic families of elements in the p-local stable homotopy groups of spheres known as the p-primary n-th Greek letter family. When $p \geq 3$ and $n = 1$, the α-family has a close relationship to special values of the Riemann zeta function by work of Adams [Ada66] and this family of elements is detected in the algebraic K-theory of the integers by Quillen [Qui72]. These results were one of the results that motivated Lichtenbaum [Lic73] to speculate that algebraic K-theory groups and special values of zeta functions should be closely related more generally. In [Lau99, Beh06, BL09], Behrens and Laures demonstrated a tight connection between the p-primary β-family for $p \geq 5$ and integral modular forms. The following result is therefore a higher chromatic height analogue of the results of Adams and Quillen and gives evidence for Question 1.1.

Theorem 1.5 ([AK18])**.** The p-primary β-family modulo (p, v_1) is detected in the iterated algebraic K-theory $\pi_* \mathrm{K}(\mathrm{K}(\mathbb{Z}))$ of the integers for $p \geq 5$.

This result built on computations in my PhD thesis [AK17], published in [AK21b] and joint work with Salch [AKS18]. In [AKS18], we provided a useful tool for computing factorization homology of commutative ring spectra. For simplicity, we state the result for THH, which is factorization homology for the circle. By a multiplicatively filtered commutative ring spectrum, I mean a sequence of spectra $\longrightarrow I_2 \longrightarrow I_1 \longrightarrow I_0$ equipped with structure maps $\rho_{i,j} \colon I_i \wedge I_j \longrightarrow I_{i+j}$ for all $i, j \geq 0$ satisfying certain coherence diagrams. There is an associated graded functor E_0^* from multiplicatively filtered commutative ring spectra to commutative ring spectra.

Theorem 1.6 ([AKS18])**.** Let I be a multiplicatively filtered commutative ring spectrum, then there is a spectral sequence

$$\mathcal{H}_* \mathrm{THH}(E_0^* I) \implies \mathcal{H}_* \mathrm{THH}(I_0)$$

for any generalized homology theory $\mathcal{H}$ and $E_0^* I$ is the associated graded of I.

In [AK21b], I used this to compute mod (p, v_1) topological Hochschild homology of the connective cover of the $K(1)$-local sphere, denoted simply j, at primes $p \geq 5$.

Theorem 1.7 ([AK21b])**.** Let $p \geq 5$. There is an isomorphism of graded $\mathbb{F}_p$-algebras

$$V(1)_* \mathrm{THH}(j) \cong H(E(\alpha_1, \lambda_1, \lambda_2); d) \otimes P(\mu_2) \otimes \Gamma(\sigma b).$$

Towards answering Question 1.3, I proved several $K(n)$-local vanishing results. There are spectra $y(n)$ that filter between $H\mathbb{F}_2$ and $\mathbb{S}$

$$\mathbb{S} = y(0) \longrightarrow y(1) \longrightarrow \cdots \longrightarrow y(\infty) = H\mathbb{F}_2.$$

We say $R \to S$ is a $K(n)$-local equivalence if the fiber F satisfies $K(n)_* F = 0$.

Theorem 1.8 ([AKQ19])**.** Whenever the map $y(n) \to H\mathbb{F}$ is a $K(m)$-local equivalence, the map $\mathrm{K}(y(n)) \longrightarrow \mathrm{K}(\mathbb{F}_2)$ is a $K(m)$-local equivalence and the map $\mathrm{TP}(y(n)) \longrightarrow \mathrm{TP}(\mathbb{F}_2)$ is a $K(m+1)$-local equivalence.

This shows that algebraic K-theory does not decrease chromatic complexity and TP increases chromatic complexity for the family of spectra $y(n)$. This result uses a technical tool proven in joint work with Salch. Using this technical result and trace methods, we also prove an upper Morava K-theory vanishing region for commutative ring spectrum forms of $BP\langle n\rangle$. Note that $L_{K(m)} BP\langle n\rangle = 0$ for $m \geq n+1$.

Theorem 1.9 ([AKS20])**.** When B is a commutative ring spectrum form of $BP\langle n\rangle$, then $L_{K(m)} K(B) = 0$ for $m \geq n+2$.

The calculations of $\mathrm{THH}_*(BP\langle n\rangle)$ for $n \geq -1$ are deep results and they are increasingly difficult as n increases. For example, Bökstedt's result $\mathrm{THH}_*(BP\langle -1\rangle) = P(\mu_0)$ [Bök87] can be used to prove Bott periodicity [HN19]. When describing the torsion in $\mathrm{THH}_*(BP\langle 1\rangle)$, Angeltveit–Hill–Lawson say it "follows a kind of tower-of-Hanoi pattern with increasingly complicated, inductively built, components" [AHL10, p.6]. In joint work with Culver and Höning, we aim to give a complete description of $\mathrm{THH}_*(\mathrm{taf}^D)$ where taf^D is an E_∞ form of $BP\langle 2\rangle$ at $p = 3$ by [HL10]. Towards this aim, we have already proven.

Theorem 1.10 ([AKCH21b]). When $m = 0, 1, 2$, there are isomorphisms

$$\mathrm{THH}_*(\mathrm{taf}^D; k(m)) \cong P(v_m) \otimes E(\lambda_1, \ldots, \lambda_{2-m}) \otimes T^n_m,$$

where T^n_m is an explicit v_m-torsion module.

In work in progress, we combine the Hochschild–May spectral sequence from Theorem 1.6. with Bockstein spectral sequences to compute $\mathrm{THH}_*(\mathrm{taf}^D; BP\langle 1\rangle)$ in [AKCH21a] where $BP\langle 1\rangle$ is explicitly constructed as the E_1 ring quotient taf^D/v_2.

1.4 Work in progress on higher Lichtenbaum–Quillen conjectures

Currently, I am working on two projects that extend the current knowledge of the red-shift conjectures. In joint work with Hahn and Wilson, we study the red-shift conjecture for Morava K-theory. Since $L_{K(m)}K(n) = 0$ if and only if $m \neq n$, one may ask whether Question 1.2 holds for $K(n)$. We show that $L_{K(n)}K(K(n)) \neq 0$, which shows that Question 1.2 does not hold for ring spectra that aren't $\mathbb{E}_2$ ring spectra, so the question only makes sense for $\mathbb{E}_n$ ring spectra for $n \geq 2$. We also note that $K(n)_{(p)} \longrightarrow L^f_n K(n)_{(p)}$ is clearly truncated, so one may ask whether $K(K(n))_{(p)} \longrightarrow L^f_{n+1}K(K(n))_{(p)}$ is truncated, meaning that the fiber has bounded above homotopy groups.

In this section, we refer to theorems from work in progress as pre-theorems. In joint work with Hahn and Wilson, we resolve Question 1.1 for the ring spectra $K(n)$.

Pre-theorem 1.1 ([AKHW21]). The following hold for $\mathrm{K}(K(n))$:

1. There are equalities $L_{K(m)}\,\mathrm{K}(K(n)) = 0$ if and only if $m \in \{0, n, n+1\}$.
2. The map $\mathrm{K}(K(n))_{(p)} \longrightarrow L^f_{n+1}\,\mathrm{K}(K(n))_{(p)}$ is truncated.

In cutting edge work [HW20], Hahn–Wilson answer Question 1.1 for truncated Brown–Peterson spectra $BP\langle n\rangle$ and our work builds on this result. It gives the first example of a positive answer to Question 1.1 for ring spectra that are not $\mathbb{E}_2$ ring spectra at all chromatic complexities.

On the way to proving their result, Hahn–Wilson prove in [HW20] that $BP\langle n\rangle$ exists as an $\mathbb{E}_3$ ring spectrum. This has made it possible to make most of the computational approach of [AR02] work for the $\mathbb{E}_3$ ring spectrum $BP\langle 2\rangle$ at $p \geq 11$. In joint work with Ausoni, Culver, Höning, and Rognes, we give an explicit computation of mod (p, v_1, v_2)-homotopy of $\mathrm{K}(BP\langle 2\rangle)$.

Pre-theorem 1.2 ([AKAC+21]). Algebraic K-theory of the $\mathbb{E}_3$ ring spectrum $BP\langle 2\rangle$ has pure FP-type 3 at $p \geq 11$.

This resolves Question 1.4 for $BP\langle 2\rangle$. In particular, this gives an explicit description of conjectural étale cohomology of $BP\langle 2\rangle$, which speculatively could be used to relate algebraic K-theory of $BP\langle 2\rangle$ to automorphic forms in the spirit of Question 1.1.

2 Equivariant topology and geometric applications

This section is split into four sub-sections: Section 2.1 a brief survey of terms in equivariant topology, Section 2.2 on the Segal conjecture for topological Hochschild homology, Section 2.3 on combinatorial equivariant topological Hochschild homology, and Section 2.4 on work in progress in synthetic motivic homotopy theory and equivariant topology.

2.1 Equivariant algebra and topology

The field of equivariant topology was revitalized in the last decade with the groundbreaking work of Hill, Hopkins, and Ravenel [HHR16] resolving almost all remaining cases of the Arf–Kervaire invariant one problem. Since this work, there has been significant advances in the subject building on the machinery in their paper. One of the foundational parts of their work is the norm functor,

$$N_H^G \colon \mathsf{Sp}^H \longrightarrow \mathsf{Sp}^G,$$

which they define for any finite group G. When G is a compact Lie group these constructions were not available, until Angeltveit–Blumberg–Gerhardt–Hill–Lawson–Mandell [ABG+18] gave a construction of norms $N_{\mu_n}^{\mathbb{T}}$ from the group of n-th roots of unity in the circle $\mathbb{T}$, regarded as the group unit vectors in the complex numbers.

The work of [HHR16] also relied on the theory of genuine commutative G ring spectrum. One may ask what the associative analogue of a genuine commutative G-ring spectrum is and one answer in the case when G is the cyclic group of order 2 is a ring spectrum with anti-involution. Ring spectra with anti-involution play key role in surgery theory as they are the input for real algebraic K-theory, which is a genuine equivariant version of algebraic K-theory that receives a ring spectrum with anti-involution as input and recovers various forms of L-theory. Just as classically algebraic K-theory can be approximated by topological Hochschild homology, Real algebraic K-theory can be approximated by Real topological Hochschild homology. One may then form Real analogues of topological cyclic homology and related invariants and use them to approximate Real algebraic K-theory.

My work in equivariant homotopy theory falls into four projects: The study of the Segal conjecture for topological Hochschild homology of ring spectra, the study of norms for the group $O(2)$ of two-by two orthogonal matrices and equivariant Witt vectors, generalizations of real topological cyclic homology to other groups of equivariance, and new perspectives of equivariant homotopy theory for odd order cyclic groups.

2.2 The Segal conjecture

Algebraic K-theory of the sphere spectrum $\mathrm{K}(\mathbb{S})$ has important geometric applications. In particular, it splits as

$$\mathrm{K}(\mathbb{S}) = \mathbb{S} \oplus \mathrm{Wh}^{\mathrm{DIFF}}(*) \tag{1}$$

where $\mathrm{Wh}^{\mathrm{DIFF}}(*)$ is a delooping of stable h-cobordism space $\mathcal{H}(*)$. Computing the algebraic K-theory groups of the sphere spectrum is therefore highly desirable.

There is a family of ring spectra $X(n)$ that filter between the sphere spectrum and complex cobordism spectrum

$$\mathbb{S} = X(0) \longrightarrow X(1) \longrightarrow \cdots \longrightarrow X(\infty) = \mathrm{MU}\,.$$

Dundas–Rognes [DR17] suggested a program for computing algebraic K-theory of $\mathbb{S}$ by descent along the map of ring spectra $\mathbb{S} \to X(n)$ so the algebraic K-theory of $X(n)$ gives a first approximation to algebraic K-theory of the sphere spectrum.

The proof of the Segal conjecture [Car84] was one of the most important advances in equivariant topology in the 1980's. This conjecture was generalized to topological Hochschild homology of ring spectra by Lunøe-Nielsen–Rognes [LNR11]. To compute algebraic K-theory, one of the key steps in the proof is a weak form of the Segal conjecture. In joint work with Quigley [AKQ19], I prove a strong form of the Segal conjecture for THH of the family of spectra $X(n)$.

Theorem 2.1 ([AKQ19])**.** The Tate valued Frobenius map

$$\varphi_p \colon \ \mathrm{THH}(X(n)) \longrightarrow \mathrm{THH}(X(n))^{tC_p}$$

is an equivalence after p-completion for all primes p.

This result is interesting in its own right because it is a generalized version of the classical Segal conjecture and it also makes computations of $X(n)$ more accessible, giving a potential approach to algebraic K-theory of the sphere spectrum.

2.3 Combinatorial equivariant factorization homology

In [ABG$^+$18], they define norms $N_{\mu_n}^{\mathbb{T}}(R)$ from the groups of unity to the circle group $\mathbb{T}$. In [BGHL18], the authors then used this theory to define a new theory of Witt vectors and Hochschild homology for associative μ_n-Green functors.

In joint work with Gerhardt and Hill [AKGH21], I define the norm $N_{D_2}^{O(2)}(R)$ for a ring spectrum with anti-involution R, where D_2 is the cyclic group of order 2. To prove that our definition is reasonable, we prove a fundamental property of the norm functor.

Theorem 2.2 ([AKGH21])**.** The restriction of the functor $N_{D_2}^{O(2)}$ to the category of genuine commutative D_2-spectra

$$N_{D_2}^{O(2)} \colon \ \mathrm{CAlg}(\mathsf{Sp}^{D_2}) \longrightarrow \mathrm{CAlg}(\mathsf{Sp}^{O(2)})$$

is left adjoint to the restriction functor $\iota_{D_2}^*$.

Another property of the norm for finite groups is the multiplicative double coset formula. For our new notion of norm $N_{D_2}^{O(2)}$, we also prove a such a formula, extending [DMPP17].

Theorem 2.3 ([AKGH21])**.** Under cofibrancy hypotheses on R, there is an equivalence of D_{2m}-spectra

$$\iota_{D_{2m}}^* N_{D_2}^{O(2)}(R) \simeq N_{D_2}^{D_{2m}} R \wedge^L_{N_e^{D_{2m}} \iota_e^* R} N_{\triangle}^{D_{2m}} \iota_{\triangle}^* c_\zeta R$$

where ζ is the generator of $\mu_{2m} \subset D_{4m}$ and $\triangle = \zeta D_2 \zeta^{-1}$.

We use these results to motivate a new definition of Real Hochschild homology

$$\underline{\mathrm{HR}}_*^{D_{2m}}(\underline{M})$$

of a ring with anti-involution $\underline{M}$. We prove that this construction has structure maps

$$R\colon \underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M})^{\mu_{p^k}} \longrightarrow \underline{\mathrm{HR}}_0^{D_{2p^{k-1}}}(\underline{M})^{\mu_{p^{k-1}}}.$$

as well as Frobenius and Verschiebung maps F and V coming from the equivariant structure. Therefore, we can define

$$\mathbb{W}(\underline{M};p) = \lim_R \underline{\mathrm{HR}}_0^{D_{2p^k}}(\underline{M})^{\mu_{p^k}}$$

and prove the following refinement of a result of Hesselholt [Hes97], where we write TCR for Real topological cyclic homology.

Theorem 2.4 ([AKGH21]). Given a ring spectrum R with anti-involution such that a certain $R^1 \lim$ vanishes

$$\mathrm{TCR}_{-1}(R) = \mathbb{W}(\underline{\pi}_0^{D_2} R; p)_F$$

where $\mathbb{W}(\underline{M};p)_F$ is the coinvariants of an operator F on $\mathbb{W}(\underline{M};p)$.

In joint work with Peroux and Merling [AKMP21], we extend the theory of combinatorial equivariant factorization homology to more general compact Lie groups. In particular, a crossed simplicial group (csg), denoted $\Delta\mathfrak{G}$ generalize both simplicial groups and Connes' cyclic category Λ. We have several constructions (a Bökstedt model, a norm model, and an ∞-categorical model) of topological csg homology of a ring spectrum with anti-involution in $\mathfrak{G}_0$-spectra

$$N_{\mathfrak{G}_0}^{\mathfrak{G}}(R) = \mathrm{TH}\,\mathfrak{G}(R)$$

and consequently we give new combinatorial models for norm functors for compact topological groups. These extend the algebraic constructions of Loday and Fiedorowicz [FL91] known as the homology of crossed simplicial groups.

Of particular interest is the case of the quaternionic crossed simplicial group. In this case, let R be an Q_4 ring spectrum with anti-involution, where Q_4 is the cyclic group of order 4. A special case of our construction gives a combinatorial model for the norm

$$N_{Q_4}^{\mathrm{Pin}_-(2)}(R) = \mathrm{THQ}(R),$$

which we aim to use to better understand involutive knot Floer homology in analogy with the relationship between topological Hochschild homology and knot Floer homology.

Goal 2.5. Construct a Hodge-to-de Rham spectral sequence for THQ that can be used to study knot Floer homology.

As a completely distinct question, this new theory has potential to receive a natural trace map from equivariant algebraic K-theory.

Goal 2.6. Construct a trace map

$$\mathrm{K}_G(R) \longrightarrow N_G^{G\times\mathbb{T}}(R).$$

To give a further refinement of the trace map, we aim to extend the theory of cyclotomic spectra to this level of generality.

Goal 2.7. Define $\mathfrak{G}$-cyclotomic structure so that there is a notion of topological $\mathfrak{G}$-cyclic homology

$$\mathrm{TC}\,\mathfrak{G}(R).$$

When $\Delta\mathfrak{G} = \Lambda \times \{G\}$ is the trivial extension of Connes' cyclic category Λ by the crossed simplicial group associated to the constant simplicial group G, we call this TC_G. This theory should refine the cyclotomic trace

$$K_G(R) \longrightarrow \mathrm{TC}_G(R) \to N_G^{G\times\mathbb{T}}(R)$$

and when $\Delta\mathfrak{G}$ is the dihedral category this recovers TCR.

2.4 Synthetic C_p-equivariant homotopy

One can use motivic homotopy theory $\mathrm{SH}(\mathbb{R})$ over $\mathrm{spec}(\mathbb{R})$ to simplify computations in Sp^{D_2}. In the case of a cyclic group of order p, denoted C_p, where p is odd there is not a known good replacement for $\mathrm{SH}(\mathbb{R})$. In work in progress with Behrens, Belmont, and Kong [AKBBK21], I aim to give a purely synthetic construction equipped with a realization functor to Sp^{C_p}. Synthetic homotopy theory was initiated by [GIKR18] and extended significantly by [Pst18]. Recently, this was used by Burkland–Hahn–Senger [BHS20] to extend the cellular $\mathbb{R}$-motivic stable homotopy category in a way that allows for realization functors recovering $RO(C_2)$-graded homotopy groups of C_2-spectra.

Goal 2.8. Combine the theory of recollement in stable homotopy theory and synthetic spectra to give an artificial anologue of $\mathrm{SH}(\mathbb{R})$ equipped with a realization functor to Sp^{C_p} for odd primes p.

Ravenel has suggested that a nice C_p-equivariant Brown–Peterson spectrum BP could be used to resolve the remaining case of the odd primary Arf–Kervaire invariant one problem in analogy with the solution of the 2-primary Kervaire invariant one problem by [HHR16]. The key part of this being the existence of a nice Adams–Novikov spectral sequence based at this C_p-equivariant BP. Some progress in this direction appears in work of Hahn–Senger–Wilson [HSW20]. In our work, we skip the construction of BP and instead aim to construct an analogous spectral sequence.

Goal 2.9. Resolve the 3-primary Arf–Kervaire invariant one problem using our new artificial 3-primary analogue of $\mathrm{SH}(\mathbb{R})$.

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