Research Statement

Gabe Angelini-Knoll

I am an algebraic topologist interested in computations of algebraic K-theory and related invariants. Algebraic K-theory takes a ring R as input and produces an abelian group $\mathrm{K}_m(R)$ for each natural number m. These groups encode deep arithmetic and geometric information and they have numerous applications. For example, Voevodsky-Rost's proof [45] of the Lichtenbaum-Quillen conjecture (LQC) implies that algebraic K-theory groups recover special values of zeta functions. The focal point of my research is the interaction between the arithmetic of rings encoded in algebraic K-theory and the fields of equivariant and chromatic homotopy theory.

Chromatic homotopy theory is the study of periodic phenomena in the homotopy groups of spheres. The homotopy groups of spheres encode maps between spheres up to continuous deformation. They are important because spheres are the basic building blocks of nice topological spaces and maps between spheres are the gluing data for building more complicated topological spaces. In the 1960's, Adams [1] first showed that Bott periodicity in the homotopy groups of the stable orthogonal group can be used to construct a periodic family of elements in the stable homotopy groups of spheres called the α-family. Smith [43] constructed a periodic family of longer wavelength called the β-family and Miller-Ravenel-Wilson [35] generalized these families of elements to periodic families of chromatic complexity n called the n-th Greek letter families. These periodic families have deep connections to arithmetic, for example the α-family encodes special values of the Riemann zeta function and the periodic families of higher height are related to modular forms and automorphic forms.

In the 1970's, Quillen [39] invented higher algebraic K-theory generalizing the Picard group, the group of units, and the Brauer group of a ring. As a first computation, he computed higher algebraic K-theory groups of finite fields [38]. As a consequence of his work, the α-family is detected in the algebraic K-theory groups of finite fields and these groups also encode special values of the Riemann zeta function. Modern Algebraic K-theory allows for generalizations of rings as input, such as ring spectra that arise in algebraic topology. This allows one to iterate the algebraic K-theory construction. My work sheds light on a highly regarded open conjecture in the intersection of chromatic homotopy theory and algebraic K-theory called the Ausoni-Rognes red-shift conjecture [11]. This conjecture generalizes the LQC to higher chromatic heights and it suggests a deep connection between periodicity in the homotopy groups of spheres and periodicity in algebraic K-theory and related invariants. In particular, the philosophy of the red-shift conjecture is that iterating the algebraic K-theory construction increases the chromatic complexity of the periodic information that is detected.

In my work, I compute algebraic K-theory of ring spectra and related invariants. I take an approach known as trace methods, where one computes algebraic K-theory by successive approximations. The first approximation is topological Hochschild homology (THH), which also has interesting applications to deformation theory, obstruction theory, and string topology. There is a closer approximation to algebraic K-theory that is built out of the extra structure on THH called topological cyclic homology (TC). This invariant and its relatives also have interesting applications outside of computing algebraic K-theory, for example recent groundbreaking work of Bhatt-Morrow-Scholze [12] shows that one can build a new cohomology theory out of them that recovers étale cohomology, crystalline cohomology, and de Rahm cohomology.

There is also a more recent theory of Real algebraic K-theory of rings R with anti-involution, due to Hesselholt-Madsen [23], whose output is an abelian group $\mathrm{KR}_\alpha(R)$ for each $\mathbb{Z}/2$-representation α. Real algebraic K-theory has its antecedents in Karoubi-Villamayor's work on Hermitian K-theory and Atiyah's work on Real topological algebraic K-theory, and the capitalization of Real is Atiyah's convention [10, 28]. To make this invariant more computable it is important to develop the theory of trace methods in this setting and there has been significant progress on this goal by Dotto [19], Högenhaven [25], and Dotto-Patchkoria-Moi-Reeh [18] building on Hesselholt-Madsen [23]. This is a relatively new area of research and there are still many unanswered questions. My work will further develop the foundations of the Real analogue of topological Hochschild homology and topological cyclic homology and contribute new computations of Real topological Hochschild homology. A long term goal of this project is to understand how exotic periodic phenomena in the $\mathbb{Z}/2$-equivariant homotopy groups of spheres interacts with Real algebraic K-theory.

This document is divided into three sections: a brief section introducing trace methods, a section on completed work, and a section on work in progress and future directions.

1 Background on Trace Methods

In the early 1990's, Bökstedt-Hsiang-Madsen [14] developed a technique for computing algebraic K-theory called trace methods, which relies on tools from algebraic topology. In particular, algebraic K-theory of ring spectra can be approximated by topological Hochschild homology (THH), which may be built in a similar fashion to Hochschild homology (HH) of associative algebras by working over the deeper base of the sphere spectrum. In particular, there is a highly nontrivial map called the Bökstedt trace $\mathrm{K}_*(A) \to \mathrm{THH}_*(A)$, which refines the Dennis trace map to Hochschild homology. Moreover, topological Hochschild homology has the structure of a cyclotomic spectrum which includes an action of the circle group S^1 along with structure maps called the Tate valued Frobenius maps, following [36]. This extra structure allows one to build a further refinement of topological Hochschild homology called topological cyclic homology (TC). For a large class of ring spectra A, the groups $\mathrm{K}_n(A)$ and $\mathrm{TC}_n(A)$ agree after completing at a prime p for $n \geq 0$ by [22, 21]. Recent work of Nikolaus-Scholze [36], simplifies the definition of topological cyclic homology of connective ring spectra as the fiber in the fiber sequence

$$\mathrm{TC}(A)_p \to \mathrm{TC}^-(A)_p \xrightarrow{\psi_p^{hS^1} - \mathrm{can}} \mathrm{TP}(A)_p \tag{1}$$

where the map $\psi_p^{hS^1} - \mathrm{can}$ is part of the data of the cyclotomic structure. More precisely, $\mathrm{TC}^-(A)$, known as topological negative cyclic homology, is the S^1-homotopy fixed points of $\mathrm{THH}(A)$ and $\mathrm{TP}(A)$, known as topological periodic cyclic homology, is the S^1-Tate construction of $\mathrm{THH}(A)$. The invariants THH, TC^-, TP, and TC have proven to be interesting in their own right as well, by recent work of B. Bhatt, M. Morrow, and P. Scholze [12].

2 Completed Work

2.1 The THH-May spectral sequence

Multiplicative filtrations of rings are ubiquitous in algebra, but in homotopy theory, these constructions have been less accessible. A simple reason is that ideals in ring theory are understandable, but in ring spectra they are more complicated. J. Smith suggested a notion of ideal of a ring spectrum, now called Smith ideals and the theory was further developed by Hovey in [27]. In [7], A. Salch and I develop a theory of multiplicatively filtered commutative ring spectra that generalize Smith ideals and are a useful notion for generalizing the flavor of filtering by powers of an ideal. In [7], we construct a large class of examples of such filtrations using the idea of the Whitehead tower from algebraic topology.

Theorem 2.1 (Angelini-Knoll, Salch [7])**.** There is an explicit model for a mulitplicative Whitehead filtration

$$\dots \tau_{\geq 3}R \to \tau_{\geq 2}R \to \tau_{\geq 1}R \to R$$

of a connective commutative ring spectrum R equipped with structure maps

$$\rho_{i,j} \colon \tau_{\geq i}R \wedge \tau_{\geq j}R \to \tau_{\geq i+j}R$$

satisfying commutativity, associativity, unitality, and compatibility axioms.

Here $\tau_{\geq n}R$ is built from R by killing homotopy groups below n in a structured way. The associated graded commutative ring spectrum of this filtration, which can be described additively as

$$E_0\tau_{\geq \bullet}R = \bigvee_{i\geq 0} \tau_{\geq i}R/\tau_{\geq i+1}R$$

is weakly equivalent to $H\pi_*R$, or in other words the generalized Eilenberg-MacLane spectrum associated to the graded ring π_*R. In the special case of connective topological K-theory ku whose homotopy groups are $\pi_*ku \cong \mathbb{Z}[\beta]$ where β is the Bott element in degree 2, the filtration exactly mimics filtering by powers of the ideal generated by β in algebra.

In [7, Thm. 3.4.8], we then prove that there is a spectral sequence in topological Hochschild homology associated to a general filtered commutative ring spectrum. Our spectral sequence was motivated by May's spectral sequence from [33] where he uses a filtration of a Hopf algebra by powers of the augmentation ideal to filter the bar construction and produce a spectral sequence computing the cohomology of that Hopf algebra. Here, I will state a special case of the theorem as stated in [7]

Theorem 2.2 (Angelini-Knoll, Salch [7])**.** There is a May-type spectral sequence of the form

$$E_{*,*}\,\mathrm{THH}(H\pi_*R) \Rightarrow E_*\,\mathrm{THH}(H\pi_*R).$$

for any connective homology theory E, where the second grading on the input is the one coming the May filtration.

In fact the theorem is much more general. By [34], when R is a commutative ring spectrum $THH(R)$ may be constructed as the tensoring $S^1 \otimes R$ of R with S^1 in the category commutative ring spectra. Our spectral sequence also applies to tensoring with any simplicial finite set $X_\bullet$ producing a spectral sequence

$$E_{*,*}(X_\bullet \otimes E_0 I_\bullet) \Rightarrow E_*(X_\bullet \otimes I_0)$$

where $I_\bullet$ is a general multiplicative filtered commutative ring spectrum. Therefore, we produce a spectral sequence in higher topological Hochschild homology as well.

2.2 Topological Hochschild homology of algebraic K-theory of finite fields

In [3], I used the spectral sequence of Theorem 2.2 to compute mod (p, v_1)-homotopy of the topological Hochschild homology of the algebraic K-theory spectrum of a large class of finite fields. In particular, when q is a prime power that generates $\mathbb{Z}/p^2$, the algebraic K-theory spectrum $\mathrm{K}(\mathbb{F}_q)$ is weakly equivalent to the p-complete image of J spectrum j after p-completion. It is also weakly equivalent to the connective cover of the localization of the sphere spectrum at the first Morava K-theory $K(1)$. This spectrum may be considered as a ring spectrum of chromatic complexity one because it detects the α-family and not the n-th Greek letter family for $n > 1$. The mod (p, v_1)-algebraic K-theory of the image of J, therefore sheds light on the chromatic red-shift conjecture of Ausoni-Rognes.

Theorem 2.3 (Angelini-Knoll [3])**.** There is an isomorphism of graded rings

$$V(1)_*\,\mathrm{THH}(j) \cong P(\mu) \otimes \Gamma(\sigma b) \otimes H_*(E(\alpha_1, \lambda_1', \lambda_2); d(\lambda_2) = \alpha_1\lambda_1').$$

for $p \geq 3$.

Here the notation $H_*(M; d)$ means the homology of the differential graded algebra M modulo the differential d.

2.3 Detecting the β-family in iterated algebraic K-theory of finite fields

In [6], I built on computations Theorem 2.3 and proved that periodicity of higher chromatic complexity is detected in iterated algebraic K-theory of finite fields, or in other words, algebraic K-theory of j.

There is a periodic family of height 2 in the homotopy groups of spheres called the β-family, which was first constructed and proven to be nontrivial by L. Smith [43]. This family is part of an infinite number of Greek letter families that are intimately connected to chromatic complexity. In particular, the first Greek letter family is the α-family and the second Greek letter family is the β-family and there is potentially a nontrivial n-th Greek letter family for each positive integer n [35]. The red-shift conjecture of Ausoni-Rognes suggests that iterating algebraic K-theory should increase the wavelength of periodicity that is detected. In [6], I prove this for iterated algebraic K-theory of finite fields and the integers.

Theorem 2.4 (Angelini-Knoll [6])**.** The β family is detected in iterated algebraic K-theory of finite fields.

The β family is an infinite periodic family of elements in π_*S and since the iterated algebraic K-theory of finite fields, or in other words $\mathrm{K}(j)$ is a ring spectrum, there is a unit map $\pi_*S \to \mathrm{K}_*(j)$ and the claim is that the β family is nontrivial in the image of this map. As a consequence of the method of proof of this result, we can also describe iterated algebraic K-theory of the integers.

Corollary 2.5 (Angelini-Knoll [6])**.** The β-family is detected in iterated algebraic K-theory of the integers.

This gives evidence for a version of the red-shift conjecture, which I call the Greek-letter family red-shift conjecture. For the following conjecture, we assume that at the given prime the n-th and $n+1$-st Greek letter family is known to be nontrivial.

Conjecture 2.6 (Greek letter family red-shift conjecture)**.** If R detects the n-th Greek letter family, then $\mathrm{K}(R)$ detects the $n+1$-st Greek letter family.

In the case $n = 0$, we can let $R = \mathbb{Z}_p$ and the fact that the α-family is detected in $\mathrm{K}(\mathbb{Z}_p)$ is evidence of the conjecture above. Corollary 2.5 gives evidence for the conjecture when $n = 1$.

The algebraic K-theory groups of the integers also encodes special values of the Riemann zeta function and there is a dictionary between chromatic height one information and these special values. There is also a dictionary between chromatic height 2 information and modular forms. In particular, Behrens proved that the (divided) β-family corresponds to modular forms satisfying certain congruences. Our Theorem [6] therefore proves that iterated algebraic K-theory groups of the integers correspond to certain modular forms, which may be considered as a higher chromatic height analogue of the Lichtenbaum-Quillen conjecture.

2.4 The Segal conjecture for THH of Ravenel spectra

In the late 1970's, Segal conjectured that there is a close relationship between the cohomotopy of the classifying space of a finite group G and the Burnside ring of the group G after suitable completion. The Segal conjecture may be extended to the question of whether the map $S^G \to S^{hG}$ is a weak equivalence after a certain completion. On π_0 this recovers Segal's original conjecture. In the case $G = C_p$, the completion is simply p-completion and the conjecture was resolved by Lin [29] at $p = 2$ and Gunawardena [2] at odd primes. Note that using the isotropy separation diagram, the map $S^{C_p} \to S^{hC_p}$ is a weak equivalence if and only if the map $S \to S^{tC_p}$ is a weak equivalence, where $(-)^{tC_p}$ is the C_p-Tate construction of S, which arises as the cofiber of the norm map from C_p-homotopy orbits to C_p-homotopy fixed points.

Since $THH(S) \simeq S$ as C_p-spectra, the Segal conjecture can generalized to the Segal conjecture for topological Hochschild homology. Lunøe-Nielsen-Rognes [30] prove the topological Hochschild homology version of the Segal conjecture of MU and BP; i.e., they prove that the map

$$THH(MU) \to THH(MU)^{tC_p}$$

is an equivalence after p-completion and they prove the same statement for BP. Their method of proof uses a topological version of the Singer construction defined as $(A^{\wedge p})^{tC_p}$ in [31]. This construction has the property that $H_*\left((A^{\wedge p})^{tC_p}\right) \cong R_+H_*(A)$ where $R_+H_*(A)$ is the classical Singer construction of the comodule $H_*(A)$ over the dual Steenrod algebra. The classical Singer construction has the property that there is a map $H_*(A) \to R_+H_*(A)$ inducing an equivalence on Ext-groups in the category of comodules over the dual Steenrod algebra; ie, an Ext-equivalence.

In joint work with J.D. Quigley, I extended these results to the family of Thom spectra $X(n)$ which were constructed by Ravenel [40] and interpolate between the sphere spectrum and MU

$$S = X(1) \to X(2) \to \ldots \to X(\infty) = MU.$$

These spectra played a crucial role in the famous proof of the Ravenel conjectures due to Devniatz-Hopkins-Smith [17, 26]. They are also useful for approximating the sphere spectrum using descent along the map $S \to X(n)$. In particular, Dundas-Rognes [20] show that using descent techniques along the map $S \to X(n)$, algebraic K-theory of $X(n)$ is a first approximation to algebraic K-theory of the sphere spectrum. Since algebraic K-theory of the sphere spectrum encodes information about stable diffeomorphisms of manifolds and is a fundamental object in derived algebraic geometry, any information about algebraic K-theory of the sphere spectrum is highly desirable. In [4], J.D. Quigley and I prove a result that is a first approximation to algebraic K-theory of $X(n)$.

Theorem 2.7 (Theorem 5.4 [4])**.** There is an equivalence after p completion

$$\mathrm{THH}(X(n)) \xrightarrow{\simeq_p} \mathrm{THH}(X(n))^{tC_p}.$$

As a consequence of our result and Tsalidis' theorem [44, 15], we have the following corollary.

Corollary 2.8. There is an equivalence after p completion

$$TC^-(X(n)) \xrightarrow{\simeq_p} TP(X(n)).$$

This significantly simplifies the computation of $TC(X(n))_p$ using the Nikolaus-Scholze [36] fiber sequence (1).

3 Current Projects and Future Directions

3.1 Chromatic complexity of algebraic K-theory of $y(n)$

There is a family of cohomology theories called Morava K-theory, which are useful for detecting chromatic height. In particular, if X is a spectrum, we say X has type n if $K(n)_*X \ncong 0$, but $K(i)_*X \cong 0$ for $i < n$. There are a family of Thom spectra due to Mahowald-Ravenel-Shick [32] that can be constructed using the Thom construction associated to a certain filtration of spaces

$$* \to \Omega J_{p-1}(S^2) \to \Omega J_{p^2-1}(S^2) \to \ldots \to \Omega J(S^2)$$

where $J(S^2)$ is the James construction on the 2-sphere, which comes equipped with a natural filtration and has the property that $J(S^2) \simeq \Omega\Sigma S^2$. This produces the family of spectra

$$S \to y(1) \to y(2) \to \ldots \to y(\infty) = H\mathbb{F}_p$$

where the fact that $y(\infty) = H\mathbb{F}_p$ is a result of Mahowald at the prime 2 and a folklore result of Hopkins at odd primes. These spectra have type n in the sense described above. In work in progress [5], J.D. Quigley and I compute the type of the relative algebraic K-theory of $y(n)$. In particular, write $\mathrm{K}(y(n) \to H\mathbb{F}_p)$ for the fiber of the map $\mathrm{K}(y(n)) \to \mathrm{K}(H\mathbb{F}_p)$.

Theorem 3.1 (Angelini-Knoll,Quigley [5])**.** The relative algebraic K-theory spectrum

$$\mathrm{K}(y(n) \to H\mathbb{F}_p)$$

has type at least $n+1$.

This result is in the same spirit of the red-shift conjecture, since we show that if the fiber of a map of ring spectra $R \to S$ has type n, then $\mathrm{K}(R \to S)$ has type $n+1$ in an important family of examples. In order to prove this result, we compute Morava K-theory of topological negative cyclic homology and topological periodic cyclic homology of the spectra $y(n)$. This is accomplished using the homological homotopy fixed point spectral sequence pioneered by work of Bruner-Rognes [16] along with Margolis homology computations. This theorem also relies on work in progress of Salch [41], which shows that Morava K-theory $K(n)_*$ commutes with certain cofiltered limits for $n \geq 1$.

3.2 Topological Hochschild homology of topological automorphic forms

The Brown-Peterson spectrum BP has been used heavily in chromatic homotopy theory to make new computations of the homotopy groups of spheres and draw connections between chromatic height and the height of p-typical formal groups. The homotopy groups of BP are a polynomial algebra over $\mathbb{Z}_{(p)}$ on generators v_n for $n \geq 1$. These generators v_n are intimately connected to periodicity of height n. By coning off generators v_n for $n > 2$, we can form $BP\langle 2\rangle$ whose homotopy groups are polynomial over $\mathbb{Z}_{(p)}$ on v_1 and v_2. This spectrum detects height 2 information and therefore it can be used to verify the red-shift conjecture from $n = 2$ to $n = 3$. In particular, we would like to show that algebraic K-theory of $BP\langle 2\rangle$ detects height three periodicity. To approach this question it is necessary to compute topological Hochschild homology of $BP\langle 2\rangle$. M. Hill and T. Lawson [24, Thm. 4.2] show that there is a model for $BP\langle 2\rangle$ as a commutative ring spectrum at the prime 3. Their work uses the recent theory of spectra associated to Shimura curves of small descriminant, a version of topological automorphic forms. This construction presents $BP\langle 2\rangle$ as a highly structured ring spectrum and makes it more amenable to computations. Using this model, D. Culver and I are pursuing mod p topological Hochschild homology of $BP\langle 2\rangle$.

Goal 3.2. Compute mod p topological Hochschild homology of $BP\langle 2\rangle$ as a first approximation to mod p algebraic K-theory of $BP\langle 2\rangle$.

Our approach is motivated by work of Angeltveit-Hill-Lawson [9] on $THH(BP\langle 1\rangle)$, but we have the additional tool of the spectral sequence of Theorem 2.2 at our disposal. In particular, the THH-May spectral sequence

$$S/3_*\,\mathrm{THH}(H\pi_*BP\langle 2\rangle) \Rightarrow S/3_*\,\mathrm{THH}(BP\langle 2\rangle)$$

allows us to pass along the diagonal of the square of spectral sequences

$$\begin{tikzcd}
\pi_*\,\mathrm{THH}(BP\langle 2\rangle; H\mathbb{F}_3)[v_1,v_2] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \pi_*\,\mathrm{THH}(BP\langle 2\rangle; k(2))[v_1] \arrow[d, Rightarrow] \\
\pi_*\,\mathrm{THH}(BP\langle 2\rangle; k(1))[v_2] \arrow[r, Rightarrow] & \pi_*\,\mathrm{THH}(BP\langle 2\rangle; BP\langle 2\rangle/3)
\end{tikzcd}$$

where $k(i)$ is the connective cover of Morava K-theory $K(i)$ for $i = 1, 2$ and

$$\pi_*\,\mathrm{THH}(BP\langle 2\rangle; BP\langle 2\rangle/3) \cong S/3_*\,\mathrm{THH}(BP\langle 2\rangle).$$

By combining these approaches, we have already made significant progress on this computation.

3.3 Real topological Hochschild homology and Witt vectors for Tambara functors

In $\mathbb{Z}/2$-equivariant homotopy theory the analogue of an abelian group is a Mackey functor. A Mackey functor can be described by it's Lewis diagram, which is a diagram of abelian groups of the form

$$M(\mathbb{Z}/2) \underset{res}{\overset{tr}{\rightleftarrows}} M(*).$$

There is a symmetric monoidal product on Mackey functors called the box product $\Box$ and Green functors are commutative monoids in the category of Mackey functors with respect to this box product. Even though Green functors are commutative monoids with respect to the box product, they do not encode the genuine equivariant properties one desires. The right notion of genuine equivariant commutative monoid in Mackey functors is a Tambara functor, which is a Green functor with an extra structure map $N\colon M(\mathbb{Z}/2) \to M(*)$ called the norm map. The homotopy groups of a $\mathbb{Z}/2$-equivariant spectrum A form a Mackey functor and when A is a genuine commutative $\mathbb{Z}/2$-ring spectrum its homotopy groups form a Tambara functor.

These objects from pure equivariant homotopy theory play an important role in the theory of Real topological Hochschild homology, which generalizes topological Hochschild homology by replacing the cyclic bar construction with the dihedral bar construction. In particular, Real topological Hochschild homology takes a ring spectrum A with anti-involution as input and its output is an $O(2)$-spectrum $\mathrm{THR}(A)$, where $O(2) = S^1 \rtimes \mathbb{Z}/2$. Consequently, the homotopy groups $\underline{\pi}_0^{\mathbb{Z}/2}\,\mathrm{THR}(A)$ are a $\mathbb{Z}/2$-Tambara functor when A is more specifically a genuine commutative $\mathbb{Z}/2$-ring spectrum.

Hesselholt-Madsen [22] proved that there is a deep connection between the topological Hochschild homology of a commutative ring R and the Witt vectors of a commutative ring R. Recall that $\mathrm{THH}(R) = S^1 \otimes R$ and it therefore has an S^1-action by acting on the S^1-coordinate and a C_{p^n} action by restriction to the p^n-th roots of unity inside of $S^1 \subset \mathbb{C}^\times$. In [22], Hesselholt-Madsen proved that there is an isomorphism

$$\mathrm{W}_{n+1}(A) \cong \pi_0\,\mathrm{THH}(A)^{C_{p^n}}$$

where $\mathrm{W}_{n+1}(A)$ is the length $n+1$ p-typical Witt vectors of A. This isomorphism relates too important invariants of commutative rings, which a priori had no reason to be related. This connection between Witt vectors and topological Hochschild homology continues to be important today and it is in some sense at the heart of recent groundbreaking work in the intersection of homotopy theory and p-adic Hodge theory by Bhatt-Morrow-Scholze [12].

There is a recent generalization of the Hesselholt-Maden theorem to the setting of "Witt vectors for Green functors" due to Blumberg-Gerhardt-Hill-Lawson [13]. In particular, they construct a notion of Hochschild homology of Green functors $\underline{\mathrm{HH}}_*^G(M)$ where M is an H-Mackey functor, H and G are finite subgroups of S^1, and H is a subgroup of G. In previous work [8], the authors along with Angeltveit and Mandell constructed a spectrum $\mathrm{THH}_H(A)$ for an H-ring spectrum using a twisted version of the cyclic bar construction. In [13], they then show that there is an isomorphism

$$\underline{\mathrm{HH}}_0^G(\underline{\pi}_0^H A) \cong \underline{\pi}_0^G\,\mathrm{THH}_H(A)$$

where G is a finite subgroup of S^1 and H is a subgroup of G. This gives a description of topological Hochschild homology of an H-ring spectrum A in terms of an equivariant Hochschild homology construction, which they define to be the Witt vectors for Green functors. This is motivated by the fact that in the classical setting there is an isomorphism $\pi_0\,\mathrm{THH}(A) \cong \mathrm{HH}_0(A)$.

There is another recent generalization of the Hesselholt-Madsen theorem in the setting of Real topological Hochschild homology, by recent work in progress of Dotto-Moi-Patchkoria, building on the authors' work with Reeh in [18]. The spectrum $\mathrm{THR}(A)$ comes equipped with an $S^1 \rtimes \mathbb{Z}/2 = O(2)$-action and hence by restriction a $D_{p^n} = C_{p^n} \rtimes \mathbb{Z}/2$ action where D_{p^n} is the Dihedral group of order $2p^n$. In work in progress, Dotto-Patchkoria-Moi prove that there is an isomorphism between $\pi_0\,\mathrm{THR}(A)^{D_{p^n}}$

and the Witt vectors on the ring $\pi_0 \operatorname{THR}(A)^{C_2}$.

In work in progress with Gerhardt, we provide a common generalization of each of these constructions. In particular, we aim to give a description of the D_{p^n}-Mackey functor homotopy groups of $\operatorname{THR}(A)$ in terms of Witt vectors for C_2-Tambara functors. To accomplish this, we have constructed a version of Real Hochschild homology for $\mathbb{Z}/2$-Tambara functors $\underline{M}$ whose output is a D_{p^n}-Mackey functor $\underline{HR}^n_m(\underline{M})$ for each natural number m.

Goal 3.3. Prove that our construction of Real Hochschild homology of the $\mathbb{Z}/2$-Tambara functor $\underline{\pi}_0 A$ is equivalent in degree zero to $\underline{\pi}_0 THR(A)$ as D_{p^n}-Mackey functors.

This result would recover the results of Dotto-Moi-Patchkoria on D_{p^n} fixed points and provide a broader equivariant framework for their computations. This construction also likely generalizes further to a version of $\operatorname{THR}_D(A)$ relative to a finite subgroup of $O(2)$, which would generalize results of Blumberg-Gerhardt-Hill-Lawson [13].

3.4 Real topological cyclic homology

Recent work of Nikolaus-Scholze [36] simplifies the construction of topological cyclic homology by describing it as the fiber of the fiber sequence (1). In the equivariant setting, it is desirable to have some analogue of this result to streamline computations of Real topological cyclic homology. In [25], Högenhaven developed the theory of Real cyclotomic spectra along the lines of the genuine cyclotomic spectra of [22]. Work in progress of Quigley-Shah gives a definition of Real cyclotomic spectra along the lines of [36], in terms of a parametrized Tate construction $X^{t_{\mathbb{Z}/2}S^1}$. The parametrized Tate construction can be defined using the genuine $\mathbb{Z}/2$-equivariant classifying space $B_{\mathbb{Z}/2}S^1$. Classically, we may define the G-homotopy fixed points and G-homotopy orbits, for a finite group G, as the homotopy limit and homotopy colimit of a functor from the classifying space BG to spectra. One can then define the Tate construction X^{tG} as the cofiber of the norm map from the G-homotopy orbits to the G-homotopy fixed points. With the right adjustments, this also works for $G = S^1$ as well. In the $\mathbb{Z}/2$-equivariant setting, one needs to take the $\mathbb{Z}/2$-limit and $\mathbb{Z}/2$-colimit of parametrized $\mathbb{Z}/2$-functors $B_{\mathbb{Z}/2}S^1 \to \mathbb{Z}/2\text{-}\underline{\mathrm{Sp}}$, in the sense of Shah [42] to define the Tate construction $X^{t_{\mathbb{Z}/2}S^1}$. Here $\mathbb{Z}/2\text{-}\underline{\mathrm{Sp}}$ is the parametrized infinity category model for the category of genuine $\mathbb{Z}/2$-spectra.

Following Quigley-Shah, one may define a spectrum X to be Real p-cyclotomic if it is $O(2) = S^1 \rtimes \mathbb{Z}/2$-equivariant and it comes equipped with an $O(2)$-equivariant map $\psi_p \colon X \to X^{t_{\mathbb{Z}/2}C_p}$. In particular, X needs to be genuine $\mathbb{Z}/2$-equivariant and naive S^1-equivariant in compatible ways to define this, but this is significantly less structure than the structure needed for a genuine Real cyclotomic spectrum á la Högenhaven [25].

Assuming a Real cyclotomic model for Real topological Hochschild homology exists, denoted THR', one may use the Quigley-Shah definition of Real cyclotomic spectra to define a notion of p-complete TCR' as the fiber

$$\operatorname{TCR}'(A)_p \longrightarrow (\operatorname{THR}'(A)^{h_{\mathbb{Z}/2}S^1})_p \xrightarrow{(\psi_p)^{h_{\mathbb{Z}/2}S^1} - can} ((\operatorname{THR}'(A)^{t_{\mathbb{Z}/2}C_p})^{h_{\mathbb{Z}/2}S^1})_p. \tag{2}$$

Conjecturally, $\operatorname{TCR}'(A) \simeq \operatorname{TCR}(A)$ for bounded below E_σ-ring spectra A, where bounded below is suitably interpreted.

In order to complete this program, one needs a model for Real topological Hochschild homology as an object of $\underline{\mathrm{Fun}}_{\mathbb{Z}/2}(B_{\mathbb{Z}/2}S^1, \mathbb{Z}/2\text{-}\underline{\mathrm{Sp}})$ along with a $O(2)$-equivariant structure maps

$$\psi_p : \operatorname{THR}'(A) \to \operatorname{THR}'(A)^{t_{\mathbb{Z}/2}C_p}.$$

An object of $\underline{\mathrm{Fun}}_{\mathbb{Z}/2}(B_{\mathbb{Z}/2}S^1, \mathbb{Z}/2\text{-}\underline{\mathrm{Sp}})$ is both a genuine $\mathbb{Z}/2$-spectrum and a naive S^1-spectrum in compatible ways and therefore the goal is to construct such a model for $\operatorname{THR}'(A)$. J.D. Quigley and I aim to provide this necessary construction of THR'.

Goal 3.4. Construct a model for $\mathrm{THR}'(A)$ as a Real cyclotomic spectrum in the sense of Quigley-Shah and prove that it is equivalent to other models of $\mathrm{THR}'(A)$ as cyclotomic spectra.

One way to generalize the definition of THH in [36] would be to use the notion of the "paramaterized infinity operad" analogous to E_σ-operads where σ is the sign representation. Work in progress of Nardin-Shah provides such a notion. This gives an approach to constructing THR', but it remains to show that THR' is actually cyclotomic in the genuine sense of Högenhaven [25] and in the sense of Quigley-Shah.

One of the key technical parts needed for this program is to understand the Segal conjecture in this setting. One may define the topological $\mathbb{Z}/2$-Singer construction as the the Tate construction of the norm $N_{\mathbb{Z}/2}^{D_p}(A)^{t_{\mathbb{Z}/2}C_p}$. This is a $\mathbb{Z}/2$-spectrum and one would like to know the analogue of Lunoe-Nielsen and Rognes's work [31] in this setting.

Goal 3.5. Prove that there is an equivalence

$$A \to \left(N_{\mathbb{Z}/2}^{D_p}A\right)^{t_{\mathbb{Z}/2}C_p}$$

3.5 The Segal conjecture for THR of Thom spectra

In Section 2.4, we discussed Lin's proof of the Segal conjecture for the group $\mathbb{Z}/2$ [29]. In particular, Lin's theorem is the statement that the map $S \to S^{t\mathbb{Z}/2}$ is an equivalence after 2-completion. In $\mathbb{Z}/2$-equivariant homotopy theory, Quigley [37] proves the analogue of Lin's theorem, by showing that the map

$$S \to S^{t_{\mathbb{Z}/2}C_2}$$

is an equivalence after 2-completion. Here we use C_2 versus $\mathbb{Z}/2$ to distinguish between the two roles of the cyclic group of order 2.

The Segal conjecture for topological Hochschild homology also makes sense in the genuine $\mathbb{Z}/2$-equivariant context, provided there is a model for $\mathrm{THR}(A)$ as a functor $B_{\mathbb{Z}/2}C_2 \to \underline{\mathrm{Sp}}$. Affirmation of the Segal conjecture for topological Hochschild homology of $M\mathbb{R}$ would aid in computing algebraic K-theory of $M\mathbb{R}$. In Dundas-Rognes [20], they suggest a program for computing algebraic K-theory of the sphere spectrum using descent along the unit map $S \to MU$. I propose that $\mathbb{Z}/2$-equivariant versions of these descent techniques should enable computations of the algebraic K-theory of the $\mathbb{Z}/2$-equivariant sphere spectrum $S_{\mathbb{Z}/2}$ using descent along the map $S_{\mathbb{Z}/2} \to M\mathbb{R}$. Towards this program, I aim to prove the following first step.

Goal 3.6. Prove that there is an equivalence after 2-completion

$$\mathrm{THR}'(M\mathbb{R}) \overset{\simeq_2}{\to} \mathrm{THR}'(M\mathbb{R})^{t_{\mathbb{Z}/2}C_2}.$$

Once defined, the spectrum $\mathrm{THR}'(M\mathbb{R})$ has an $O(2)$-action and here $C_2 < S^1 < O(2)$ where $O(2) = S^1 \rtimes \mathbb{Z}/2$. Specifically, it has a genuine $\mathbb{Z}/2$-action compatible with the naive S^1-action. Note that Goal 3.6 implies that

$$THR(M\mathbb{R})^{h_{\mathbb{Z}/2}S^1} \simeq_2 (\mathrm{THR}'(M\mathbb{R})^{t_{\mathbb{Z}/2}C_2})^{h_{\mathbb{Z}/2}C_2},$$

which would simplify computations of TCR' from the previous section.

I plan to approach Goal 3.6 using homological techniques. More precisely, one needs to show that that the map

$$(H\underline{\mathbb{F}}_2)_*(X) \to (H\underline{\mathbb{F}}_2)_*\left((X^{\wedge 2})^{t_{\mathbb{Z}/2}C_2}\right),$$

induced by the Tate diagonal $X \to (X^{\wedge 2})^{t_{\mathbb{Z}/2}C_2}$, is an Ext-equivalence, where $H\underline{\mathbb{F}}_2$ is the Eilenberg-MacLane spectrum of the constant Mackey functor and Ext is in the category of comodules over the C_2-equivariant dual Steenrod algebra. By the $\mathbb{Z}/2$-topological Singer construction, we mean the object $(X^{\wedge 2})^{t_{\mathbb{Z}/2}C_2}$ associated to the C_2-spectrum X. The proof then should follow by a direct calculation of an Ext-equivalence between the $H\underline{\mathbb{F}}_2$-homology of the $\mathbb{Z}/2$-topological Singer construction on $\mathrm{THR}'(M\mathbb{R})$ and the homology of $\mathrm{THR}'(M\mathbb{R})^{t_{\mathbb{Z}/2}C_2}$.

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