Research Statement

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I am an algebraic topologist and my work explores the connections between algebraic topology, number theory, and geometric topology. My research program is primarily in two areas: the intersection of algebraic K-theory and chromatic homotopy theory and the intersection of equivariant topology and Hochschild homology. My work in algebraic K-theory and chromatic homotopy theory is primarily aimed towards understanding the arithmetic of ring spectra, which generalize rings. My program on equivariant topology and Hochschild homology has potential applications to geometric topology. The common thread in my research is the study of invariants of ring spectra.

One of the most highly regarded conjectures in intersection of chromatic homotopy and algebraic K-theory is the Ausoni–Rognes redshift conjecture, which generalizes the famous Lichtenbaum–Quillen conjecture. In fact, there is really an entire family of redshift conjectures and my work gives quantitative evidence for several of the redshift conjectures. I have done this by developing tools for computing Hochschild homology [AKS18], giving new computations of Hochschild homology using these tools [AK17, AK21, AKCH21], and building on these computations to give evidence [AKQ19, AKS20, AK22a, AK22b, AKAC$^+$22] for the redshift conjectures. I also have several ongoing projects extending this evidence and computational knowledge [AKHW, AKCH, AKH, AKAHR].

Equivariant topology has had a renaissance recently due to groundbreaking work of Hill–Hopkins–Ravenel on the Arf–Kervaire invariant problem in geometric topology. In a similar vein, my work in equivariant topology and Hochschild homology has potential applications to geometric topology. My work [AKQ21] uses equivariant structure on Hochschild homology to give a first approximation to the stable h-cobordism space of a point. My work [AKGH21] gives new perspectives on equivariant Hochschild homology and Witt vectors of rings with anti-involution. My work in progress [AKMP], gives a combinatorial approach to equivariant Hochschild homology. One of our goals is to study equivariant knot Floer homology using this theory. In work in progress [AKBBK], we also use deformations to study Borel G-equivariant homotopy. One goal of this program is to resolve the odd primary Arf–Kervaire invariant problem. Finally, in work in progress [AKQ] we study a motivic filtration on equivariant Hochschild homology in pursuit of understanding the arithmetic of equivariant ring spectra.

1 Algebraic K-theory and chromatic homotopy

This section consists of three subsections: Section 1.1 provides a brief survey of chromatic homotopy, Section 1.2 provides a brief survey of trace methods, and Section 1.3 describes my results and future research directions in the context of the redshift conjectures.

1.1 Chromatic homotopy theory

One of the fundamental open problems in algebraic topology is the computation of the *homotopy groups of spheres*. These groups are important because we want to classify CW complexes, which are topological spaces built out of spheres. The homotopy groups of spheres are the gluing data for assembling these nice spaces. In the 1930's, H. Freudenthal [Fre38] showed that the computation of the homotopy groups of spheres stabilize in high dimensions and this was a starting point of *stable homotopy theory*. Many problems in geometric topology, such as the study of bordism groups, are naturally stable phenomena.

Stable homotopy groups are much simpler and more accessible via successive algebraic approximations known as *spectral sequences*. One of my favourite spectral sequences was constructed by Adams in the 1950's [Ada58], known as the Adams spectral sequence. A variant of this spectral sequence, known as the Adams–Novikov spectral sequence was later constructed whose input is the cohomology of a Hopf algebroid associated to complex cobordism MU. The field of chromatic homotopy theory is rooted in work of Quillen [Qui69] from the 1960's relating this Hopf algebroid to the theory of formal groups.

Formal groups can be classified by their height in the sense that there is a height stratification of the p-localization of the moduli of formal groups. The tight connection between formal groups and complex cobordism can be combined with the Adams–Novikov spectral sequence to transfer this height stratification into the world of stable homotopy theory. In the 1970's, Miller–Ravenel–Wilson [MRW75] pioneered work on the chromatic spectral sequence, which computes the cohomology of the Hopf algebroid associated to complex cobordism and is closely tied to the height stratification of the moduli of formal groups. In the input of the chromatic spectral sequence, there are elements known as the *Greek letter family elements* depending on a non-negative integer n and a prime p For small enough n and large enough p these elements are known to produce families of elements in the stable homotopy groups of spheres known as the α-family, β-family, and γ-family corresponding to heights one, two, and three respectively. When viewed through this lens, the homotopy groups of spheres consist of elements of longer and longer wavelength and, by analogy with light spectrometry, we think of these wave lengths as corresponding to the increasing wavelengths of light in the color spectrum. This is where the term chromatic comes from in chromatic homotopy theory.

As the wavelength of periodicity increases, light moves in the red direction. This is why conjectures of Ausoni and Rognes, which are an important part of my research program, are called the *redshift conjectures*. These conjectures are part of a overarching philosophy of how invariants of ring spectra such as *algebraic K-theory*, *topological cyclic homology*, and *topological periodic cyclic homology*, increase chromatic complexity. This is exhibited by explicitly by an increase in the wavelength of these periodic classes. Though this phenomena is tantalizing to a stable homotopy theorist, the more exciting aspect of these conjectures is that they give a precise formulation of how one may study the *arithmetic of ring spectra* and this is my primary motivation.

1.2 Trace methods

In the 1970's, Quillen [Qui73] first constructed a topological space $\mathrm{K}(R)$ associated to a ring R whose homotopy groups are the algebraic K-theory groups $\mathrm{K}_*(R)$ of the ring. This construction takes linear algebraic data as input and miraculously produces deep arithmetic data as output. Since algebraic K-theory is defined quite broadly, we may also consider the algebraic K-theory of a ring spectrum R and use algebraic K-theory to study the arithmetic of R. Unfortunately for us, algebraic K-theory is notoriously difficult to compute; however, in the 1980's a new approach to computing algebraic K-theory was developed by Bökstedt inspired by ideas of Waldhausen. The first example of its power was demonstrated in groundbreaking work of Bökstedt–Hsiang–Madsen [BHM93]. In 2018, Nikolaus–Scholze [NS18] significantly simplified the story and this has lead to flood of new results.

The idea of trace methods is to compute algebraic K-theory by successive approximations. The first approximation to algebraic K-theory is (topological) Hochschild homology $\mathrm{THH}(R)$. Classically, one uses genuine equivariant homotopy theory and genuine cyclotomic structure to construct $\mathrm{TR}(R)$ from Hochschild homology and then define p-complete topological cyclic homoloy as the homotopy invariants with respect to an operator F on $\mathrm{TR}(R)$. The homotopy groups of $\mathrm{TR}(R)$ recover the de Rham–Witt complex [HM03], and are therefore of independent interest. The Dundas–Goodwillie–McCarthy theorem [DGM13] then proves that topological cyclic homology is a close approximation to algebraic K-theory. As a consequence, for nice enough ring spectra, algebraic K-theory and topological cyclic homology are equivalent after p-completion.

Groundbreaking work of Nikolaus–Scholze [NS18] simplified the computational approach to topological cyclic homology by describing $\mathrm{TC}(R)$ as a homotopy equalizer involving $\mathrm{TC}^{-}(R)$ and $\mathrm{TP}(R)$, which do not require genuine equivariant homotopy theory. For the expert, $\mathrm{TC}^{-}(R)$ is the S^1-homotopy fixed points of $\mathrm{THH}(R)$ and $\mathrm{TP}(R)$ is the S^1-Tate construction of $\mathrm{THH}(R)$ and they are therefore accessible using spectral sequences. These approximations to algebraic K-theory theory are interesting in their own right as well because of the relation to prismatic cohomology [BS19]. This gives an explicit computational approach to algebraic K-theory, which has led to numerous advances in the subject, for example [HM03, AR02, HW20, AKAC^{+}22, HRW22].

1.3 Results

As exhibited at the end of Section 1.1, the Ausoni–Rognes redshift conjectures are broadly about the arithmetic of ring spectra. In particular, these conjectures generalize famous conjectures of Lichtenbaum [Lic73] and Quillen [Qui75] from the 1970's. Many of the conjectures of Lichtenbaum and Quillen have been resolved by Voevodsky [Voe11]. A qualitative version of the redshift philosophy of Ausoni–Rognes was resolved in 2022 for all commutative ring spectra by Burkland–Schlank–Yuan [BSY22]. However, the redshift conjectures as originally stated by Ausoni–Rognes [AR08] are still open as well as a stronger version originally posed by Rognes [Rog00]. First, I pose a question, which can be viewed as a higher height analogue of Lichtenbaum's conjecture [Lic73, Conjecture 2.4] and discuss some evidence I have given towards answering this question.

1.3.1 Higher Lichtenbaum conjecture

In the late 1960's, Adams [Ada66] and Quillen [Qui72] showed a height one periodic family known as the α-family is detected in $\mathrm{K}_{4k-1}(\mathbb{Z})$. This the first evidence for a conjecture of Lichtenbaum [Lic73, Conjecture 2.4] that quotients of orders of algebraic K-theory groups

$$|\,\mathrm{K}_{4k+2}(\mathcal{O}_F)|/|\,\mathrm{K}_{4k+3}(\mathcal{O}_F)| = (-1)^s \cdot 2^r \cdot \zeta_F(-2k-1)$$

can be identified with special values of the Dedekind zeta function ζ_F of a totally real number field F with ring of integers $\mathcal{O}_F$. My work [AK22b] gives the first indication of a higher chromatic height analogue of Lichtenbaum's conjecture, which I will therefore call the *higher Lichtenbaum conjecture*. For notationaly simplicity, let $K^{(2)}(R)$ denote $K(K(R)$.

Question 1.1. [Higher Lichtenbaum conjecture] Is there a precise relation between the iterated algebraic K-theory groups $K_*^{(2)}(\mathcal{O}_F)$ and modular forms over $\mathcal{O}_F$?

Just as detecting the α-family was the first evidence [Lic73, §3] for Lichtenbaum's conjecture, we consider the detection of the β-family in iterated algebraic K-theory of the integers as a first step towards Question 1.1. Recall from Section 1.1 that there is a height two periodic family of elements in the p-local stable homotopy groups of spheres known as the β-family for large enough primes. In [Lau99, Beh06, BL09], M. Behrens and G. Laures demonstrated a tight connection between the p-primary β-family for $p \geq 5$ and integral modular forms. The following result is therefore a higher chromatic height analogue of the results of [Ada66, Qui72] that motivated [Lic73, Conjecture 2.4].

Theorem 1.2 ([AK22b])**.** *The p-primary β-family modulo (p, v_1) is detected in the iterated algebraic K-theory $\mathrm{K}_*^{(2)}(\mathbb{Z}))$ of the integers for $p \geq 5$.*

This gives the first evidence for Question 1.1. Theorem 1.2 from [AK22b] builds on computations from my PhD thesis [AK17], published in [AK21], and joint work with Salch [AKS18]. I summarize these results in the next section.

1.3.2 Telescopic complexity redshift

In joint work with Salch [AKS18], we provide a useful tool for computing Hochschild homology of commutative ring spectra. For simplicity, we state the result only for Hochshild homology (THH) though it applies to factorization homology of commutative ring spectra as well. By a *multiplicatively filtered commutative ring spectrum*, I mean a sequence of spectra

$$\longrightarrow I_2 \longrightarrow I_1 \longrightarrow I_0$$

equipped with structure maps $\rho_{i,j} \colon I_i \wedge I_j \longrightarrow I_{i+j}$ for all $i, j \geq 0$ satisfying commutativity, associativity, and unitality conditions. Let E_0^* denote the associated graded functor from multiplicatively filtered commutative ring spectra to commutative ring spectra.

Theorem 1.3 ([AKS18])**.** *Let I be a multiplicatively filtered commutative ring spectrum, then there is a spectral sequence*

$$\mathrm{THH}_*(E_0^* I) \implies \mathrm{THH}_*(I_0).$$

In [AK21], I used this spectral sequence to compute mod (p, v_1) Hochschild homology of the connective cover of the $K(1)$-local sphere, denoted simply j, at primes $p \geq 5$. This has also been computed by [Hön21] using different methods.

Theorem 1.4 ([AK21])**.** *Let $p \geq 5$. There is an isomorphism of graded $\mathbb{F}_p$-algebras*

$$V(1)_* \mathrm{THH}(j) \cong H(E(\alpha_1, \lambda_1, \lambda_2); d) \otimes P(\mu_2) \otimes \Gamma(\sigma b)$$

where $d(\lambda_2) = \alpha_1 \lambda_1'$.

Since j is the connective cover of the $K(1)$-local sphere, one could use a localization sequences to compute the algebraic K-theory of $L_{K(1)}\mathbb{S}$ from j. Similar ideas have recently been explored by [Lev22] using the (-1)-connective cover of the $K(1)$-local sphere instead. Therefore, this gives some progress on one of the most impenetrable of the Ausoni–Rognes redshift conjectures. To phrase the conjecture, note that there exists a finite p-local spectrum $F(n)$ of type n, which implies that there exists a self map v inducing an isomorphism in $K(n)_*$ whose telescope $v^{-1}F(n)$ we denote $T(n)$. We say that X has *telescopic complexity* n if the fiber of the canonical map $F(n) \otimes X \to T(n) \otimes X$ is bounded above. I will therefore refer to [AR08, Conjecture 4.3] as *telescopic complexity redshift*.

Question 1.5 (Telescopic complexity redshift)**.** reso If R has telescopic complexity n, does $K(R)$ have telescopic complexity $n+1$?

In [AR08, Conjecture 4.3], the conjecture is only stated for the $K(1)$-local sphere (and suitably Galois finite extensions). Therefore, 1.3 gives some of the first progress towards answering the question originally posed by Ausoni and Rognes.

1.3.3 Qualitative redshift

Though Question 1.5 remains unresolved, a more tractable question was posed by L. Meier and collaborators. This question qualitatively captures the redshift philosphy of Ausoni–Rognes and I therefore refer to it as *qualitative redshift*. To phrase this statement, I need a notion of chromatic complexity. When R is a commutative ring spectrum than there exists an integer n such that $K(m)_*R = 0$ for all $m \leq n$ and $K(m)_*R = 0$ for all $m > n$ by work of Hahn [Hah16]. More generally, we say that a ring spectrum has height n if

$$n = \min\{m : K(m)_*R \neq 0\}.$$

Question 1.6 (Qualitative redshift)**.** Does algebraic K-theory increase height by one?

For commutative ring spectra, Question 1.6 has now been proven by a tour de force beginning with work of [CMNN20, LMMT20] and culminating in results of [Yua21, BSY22]. We therefore know that the qualitative redshift conjecture is true for all commutative ring spectra, giving the most broad evidence for the redshift philosophy.

Question 1.6 also remains open for E_m ring spectra which are not commutative ring spectra. In [CMNN20, LMMT20], they prove a vanishing result for general E_1 ring spectra and Hahn–Wilson [HW20] prove Question 1.6 is true for truncated Brown–Peterson spectra.

My own work on this question slightly predates these other recent results. For my first result, note that there is a family of spectra $y(n)$ depending on a non-negative integer n that interpolate between $\mathbb{S}$ and $H\mathbb{F}_2$

$$\mathbb{S} = y(0) \longrightarrow y(1) \longrightarrow \cdots \longrightarrow y(\infty) = H\mathbb{F}_2.$$

These spectra were originally constructed by Mahowald [Mah79] and they played a key role in the attempted disproof of the telescope conjecture by Mahowald–Ravenel–Schick [MRS01]. These spectra are interesting from the perspective of chromatic homotopy theory because $K(m)_*y(n) \neq 0$ for $0 \leq m < n$ so they behave like the type n spectra $F(n)$ introduced earlier even though they are not finite spectra. We prove a vanishing version of Question 1.6 for $\mathrm{TP}(y(n))$.

Theorem 1.7 ([AKQ19])**.** The groups $K(m)_* \operatorname{TP}(y(n)$ vanish for $0 < m \leq n$. Moreover, the groups $K(m)_*K(y(n))$ vanish for $0 \leq m < n$.

The second statement, that $K(y(n))$ is $K(m)$ acyclic for $1 \leq m \leq n-1$, was also proven soon after as a consequence of [CMNN20] by entirely different methods.

Theorem 1.7 uses a technical result about commuting Morava K-theory with certain sequential limits, which I prove with Salch [AKS20] and will not mention explicitly here. As an application of this technical result, we also prove an upper Morava K-theory vanishing region for $BP\langle n\rangle$. Note that $K(m)_*BP\langle n\rangle = 0$ for $m \geq n+1$.

Theorem 1.8 ([AKS20])**.** When $BP\langle n\rangle$ is a commutative ring spectrum form of $BP\langle n\rangle$, then the groups $K(m)_*K(BP\langle n\rangle)$ vanish for $m \geq n+2$.

Soon after we posted our preprint, a more general result was proven by [LMMT20] using entirely complementary methods. Our work [AKS20] is different from [LMMT20] in that it also sheds light on TC^- and TP providing additional applications, such as [AKQ19].

1.3.4 Pure fp-type redshift

A strong form of the redshift conjectures was originally posed by Rognes [Rog00]. I refer to it as *pure fp-type redshift* below. To phrase the question, we need a definition. Note that since a finite spectrum $F(n)$ of type n has a periodic self map v inducing an isomorphism in $K(n)_*$, the homotopy groups $F(n)_*X$ are always a $\mathbb{F}_p[v]$-module. We say that a spectrum has *pure fp-type* n if the $\mathbb{F}_p[v]$-module $F(n)_*(X)$ is a finitely generated, free module following Rognes [Rog00] inspired by Mahowald–Rezk [MR99]. If X has pure fp-type n then X has *fp-type* n in the sense of [MR99] (cf. Section 1.3.5) and it also has *telescopic complexity* n (cf. Section 1.3.2). A commutative ring spectrum with telescopic complexity n also has height n (cf. Section 1.3.3). Therefore Question 1.9 is a stronger statement than Question 1.5, which is also a stronger statement than Question 1.6.

Question 1.9 (Pure fp-type redshift)**.** If R is a ring spectrum with pure fp-type n, then does $\mathrm{TC}(R)$ have pure fp-type $n+1$?

This was posed by Rognes based on computations with Ausoni [AR02] of $V(1)_* \operatorname{TC}(\ell)$ where $V(1) = \mathbb{S}/(p, v_1)$ is a four cell complex which has type 1. In particular, they proved that it is a finitely generated free $\mathbb{F}_p[v_2]$-module and therefore Conjecture 1.9 holds in this case. To do this, they begin by computing Hochschild homology (THH).

The truncated Brown–Peterson spectra are ring spectra of pure fp-type n, which are fundamental in chromatic homotopy theory. For small n these truncated Brown–Peterson spectra have more familiar names: by convention $BP\langle -1\rangle = H\mathbb{F}_p$, we have $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, and $BP\langle 1\rangle = \ell$, which is the Adams summand of p-local complex K-theory ku. The spectrum $BP\langle 2\rangle$ can also be constructed using versions topological modular forms and topological automorphic forms [HL10, LN12].

The calculations of $\mathrm{THH}_*(BP\langle n\rangle)$ for $n \geq -1$ are deep results and they are increasingly difficult as n increases. For example, Bökstedt's result $\mathrm{THH}_*(BP\langle -1\rangle) = P(\mu_0)$ [Bök87] can be used to prove Bott periodicity [HN19]. When describing the torsion in $\mathrm{THH}_*(BP\langle 1\rangle)$, Angeltveit–Hill–Lawson say it "follows a kind of tower-of-Hanoi pattern with increasingly complicated, inductively built, components" [AHL10, p.6]. In future work, I plan to give a complete description of $\mathrm{THH}_*(BP\langle 2\rangle)$. Towards this aim, I have already computed topological Hochschild homology with various coefficients in joint work with Culver and Höning.

Theorem 1.10 ([AKCH21])**.** When $m = 0, 1, 2$, there are isomorphisms

$$\mathrm{THH}_*(BP\langle 2\rangle; k(m)) \cong P(v_m) \otimes E(\lambda_1, \ldots, \lambda_{2-m}) \otimes T^n_m,$$

where T^n_m is an explicit v_m-torsion module.

In work in progress, I use different filtration techniques such as Theorem 1.3 and Bockstein spectral sequences to compute $\mathrm{THH}_*(BP\langle 2\rangle; \ell)$ in [AKCH].

In joint work [AKAC+22] with Ausoni, Culver, Höning, and Rognes I compute

$$V(2)_* \operatorname{TC}(BP\langle 2\rangle) \text{ and } V(2)_* \operatorname{K}(BP\langle 2\rangle)$$

at primes $p \geq 7$ explicitly in terms of generators and relations. This improves on a result from the landmark paper of Hahn–Wilson [HW20] in the case $n = 2$. Specifically, Hahn–Wilson [HW20] prove that $F(n+1)_* \operatorname{TC}(BP\langle n\rangle)$ is finite, or in other words $\operatorname{TC}(BP\langle n\rangle)$ has *fp-type* $n+1$. This also implies that algebraic K-theory of $BP\langle n\rangle$ has fp-type $n+1$ and telescopic complexity $n+1$. Our result is currently the first and only known proof of Conjecture 1.9 at higher heights than the original result of Ausoni–Rognes [AR02].

Theorem 1.11 ([AKAC+22])**.** The spectrum $\operatorname{TC}(BP\langle 2\rangle)$ has pure fp-type 3.

This answers Question 1.9 in the affirmative for $BP\langle 2\rangle$ because $BP\langle 2\rangle$ has pure fp-type 2.

1.3.5 Syntomic cohomology of Morava K-theory

Recent work of Hahn–Raksit–Wilson [HRW22] makes computations of algebraic K-theory and topological cyclic homology more accessible. In op. cit., the authors provide a motivic filtration on topological cyclic homology of commutative ring spectra, which is amenable to computation for certain ring spectra called chromatically quasisyntomic ring spectra. As an application they extend the computation of $V(1)_* \operatorname{TC}(BP\langle 1\rangle)$ to $p = 3$ and provide a much simpler approach to the computation for all primes $p \geq 3$. Even though ring spectra such as Morava K-theory are not chromatically quasisyntomic, they are approachable by similar means. This is the subject of ongoing work with Hahn and Wilson.

For chromatically quasisyntomic rings, the motivic filtrations on TC and TP due to [HRW22] extend the motivic filtration on TC and TP by [BMS19, BS19, BL22], whose associated graded are syntomic cohomology and (absolute) prismatic cohomology, respectively. One may therefore define syntomic cohomology and prismatic cohomology for ring spectra using this approach. I will take this as a definition of syntomic cohomology and prismatic cohomology for an arbitrary ring spectrum.

Goal 1.12. Compute the syntomic cohomology of connective Morava K-theory $k(n)$.

The syntomic cohomology R is also the E_2-page of a spectral sequence computing $\mathrm{TC}(R)$ and it often collapses at this page for bidegree reasons. For example, this is the case for $V(1)_* \mathrm{TC}(BP\langle 1\rangle)$ by [HRW22]. In general, there may be longer differentials in this spectral sequence. Nevertheless, there are questions that you can answer using this approach. We say that X has *fp-type* n if there exists a finite spectrum $F(n+1)$ of type $n+1$ such that $F(n+1)_*X$ is finite. The following conjecture is a slightly weaker version of Question 1.9. I will call it *fp-type redshift*.

Question 1.13 (fp-type redshift)**.** If R is a ring spectrum of fp-type n, then does $K(R)$ have fp-type $n+1$?

Note that if a spectrum has fp-type n then it has telescopic complexity n, so an affirmative answer to Question 1.13 implies an affirmative answer to Question 1.5. One of our goals is to answer both of these questions in the affirmative for all $k(n)$, the first example for associative ring spectra that are more highly structured.

Goal 1.14. Prove that $K(k(n))$ has fp-type $n+1$ and telescopic complexity $n+1$.

1.3.6 Iterated algebraic K-theory of finite fields

One of the main precursors to the Ausoni–Rognes red-shift conjectures was a program of Waldhausen [Wal84] for explicitly computing the algebraic K-theory of the sphere spectrum by using the algebraic K-theory of the chromatic tower and its connective counterpart

$$K(\mathbb{S}) \to \ldots \to K(\tau_{\geq 0} L_{E(1)}\mathbb{S}) \to K(\tau_{\geq 0} L_{E(1)}\mathbb{S}) \to K(H\mathbb{Z}_{(p)}).$$

There are also equivalences

$$j = \tau_{\geq 0} L_{K(1)}\mathbb{S} \simeq . \left(\tau_{\geq 0} L_{E(1)}\mathbb{S}\right)^{\wedge}_{p} \simeq K(\mathbb{F}_q)^{\wedge}_{p}$$

where I write j for the connective p-complete image of j and I have fixed $p \geq 3$ and q a generator of $\mathbb{Z}/p^2\mathbb{Z}^{\times}$. Therefore, computing $K(j)$ can be interpreted both as an approximation to the algebraic K-theory of the sphere spectrum and as an approximation to the iterated algebraic K-theory of the integers. The algebraic K-theory of the sphere spectrum is of interest in geometric topology because it recovers the homotopy groups of stable diffeomorphism spaces, which are of interest in geometric topology. The iterated algebraic K-theory of of the integers is also of interest from an arithmetic perspective as discussed in Section 1.3.1. One of the primary difficulties in moving forward with computations has been the appearance of the divided power algebra $\Gamma(\sigma b)$ in my computation of $V(1)_* \mathrm{THH}(j)$ in Theorem 1.4. This divided power algebra $\Gamma(\sigma b)$ over $\mathbb{F}_p$ is in fact infinitely generated as a ring making it difficult to rule out differentials in subsequent homotopy fixed point and and Tate spectral sequences using the traditional approach. A similar approach to that of [HRW22], where one resolves $\mathrm{THH}(j)$ by even cyclotomic spectra, should allow one to make progress on this question. Work in progress of Lee–Levy exemplifies this for the related commutative ring spectrum $\tau_{\geq -1} L_{K(1)}\mathbb{S}$. In joint work with Höning, we aim to use an even filtration to compute $V(1)_* K(j)$ [AKH].

Goal 1.15. Determine whether $K(j)$ has fp-tpe 2 and telescopic complexity 2.

One could use this computation together with a localisation sequence in order to resolve the redshift conjecture of Ausoni–Rognes [AR08, Conjecture 4.3], which suggests that $K(L_{K(1)}\mathbb{S})$ has telescopic complexity 2 (cf. Section 1.3.2). A similar approach is suggested in work of Levy [Lev22].

It has also been suggested by Hahn in personal communication that $V(1) \otimes K(j)$ is a good test case for resolving the telescope conjecture. We also aim to determine this for the algebraic K-theory of the image of J.

Goal 1.16. Determine whether the telescope conjecture is true for $V(1) \otimes K(j)$.

Since there are currently no counterexamples to the telescope conjecture, this would disprove the telescope conjecture once and for all.

1.3.7 G-theory

In 2012, C. Ausoni and J. Rognes [AR12] computed the algebraic K-theory of the first connective Morava K-theory $k(1)$ modulo (p, v_1). As part of this program, they proposed that one should consider localization sequences

$$K(L/p) \to K(L_p^\wedge) \to K(ffL_p^\wedge) \tag{1}$$

where the third term is defined as the cofiber of the transfer map associated to the map of ring spectra $i\colon L_p^\wedge \to L/p$ [Rog14, Definition 6.1]. This is in analogy with the classical localization sequence

$$K(\mathbb{F}_p) \to K(\mathbb{Z}_p^\wedge) \to K(\mathbb{Q}_p)$$

of Quillen [Qui73]. In particular, Rognes [Rog14] suggests that there exists a spectral sequence

$$H^{-s}_{\text{mot}}(ffL_p^\wedge; \mathbb{F}_{p^2}(t/2)) \implies V(1)_*K(ffL_p^\wedge)$$

computing the algebraic K-theory of this cofiber modulo (p, v_1).

One may consider similar questions for other elements $v_m \in \pi_*BP\langle n\rangle$, for example B. Antieau, T. Barthels, and D. Gepner [ABG18b] show that there is a localization sequence

$$K(\text{End}_{BP\langle n\rangle}(BP\langle n-1\rangle)) \to K(BP\langle n\rangle) \to K(BP\langle n\rangle[v_n^{-1}]).$$

When $n = 1$, this has an even nicer form by earlier work of Blumberg–Mandell [BM08]

$$K(H\mathbb{Z}) \to K(BP\langle 1\rangle) \to K(BP\langle 1\rangle[v_1^{-1}])$$

which compares nicely with the localization sequence above, but this fails for larger n. Nonetheless, these localization sequences were used to show that $K(m)_*K(BP\langle n\rangle[v_n^{-1}]) = 0$ for $m \geq n + 2$ [LMMT20, Corollary D].

Algebraically, there are two different notions of algebraic K-theory commonly considered known as K-theory and G-theory. K-theory is defined using finitely generated projective modules where as G-theory is defined using pseudo-coherent modules. For regular Noetherian rings, these two notions agree.

For ring spectra, we still do not have a good notion of G-theory even though algebraic K-theory is defined quite broadly. One reason is that the right notion of pseudo-coherent modules has not been developed. Another wrinkle in this story is that the algebraic K-theory of the fraction field $ffL_p^\wedge$ may not actually be the algebraic K-theory of a ring spectrum, but rather a ring spectrum with many objects. This makes the theory a bit more complicated. Nonetheless, we soldier on.

In joint work in progress with Ausoni, Höning, and Rognes, we aim to develop the theory of G-theory for ring spectra equipped with localization sequences such as (1). In fact, one needs to develop this theory in a way that is flexible enough to contain ring spectra with many objects. Ultimately, one may ambitiously hope to realize the algebraic K-theory of a certain maximal Galois extension of ℓ as a close approximation to Lubin–Tate E-theory E_2 as conjectured in [AR08, Conjecture 4.4].

2 Equivariant topology and Hochschild homology

This section consists of two subsections: Section 2.1 gives some background on equivariant topology and Section 2.2 describes some of my results and future work in this area.

2.1 Equivariant algebra and topology

The field of equivariant topology was revitalized in the last decade with the groundbreaking work of Hill, Hopkins, and Ravenel [HHR16] resolving almost all remaining cases of the Arf–Kervaire invariant problem. Since this work, there has been significant advances in the subject building on the machinery in their paper. One of the foundational parts of their work is the *norm functor*

$$N_H^G \colon \mathsf{Sp}^H \longrightarrow \mathsf{Sp}^G$$

from genuine H-spectra to genuine G-spectra, which they define for any finite group G.

The work of [HHR16] also relies on the theory of *genuine commutative G-ring spectra*. One may ask what the associative analogue of a genuine commutative G-ring spectrum is and one answer, in the case when G is the cyclic group of order 2, is a *ring spectrum with anti-involution*. Ring spectra with anti-involution play a key role in surgery theory as they are the input for *real algebraic K-theory*. Real algebraic K-theory is a genuine equivariant version of algebraic K-theory, which recovers various forms of L-theory and Grothendieck–Witt theory. Just as classically algebraic K-theory can be approximated by topological Hochschild homology, real algebraic K-theory can be approximated by *real (topological) Hochschild homology* [HNS22]. One may then form real analogues of topological cyclic homology and related invariants [QS21] and use them to approximate real algebraic K-theory.

2.2 Results

My results in the area of equivariant topology and Hochschild homology roughly fall into two categories: a program for understanding the arithmetic of equivariant ring spectra and a program for studying geometric topology using equivariant Hochschild homology and deformations. The first program is discussed in Section 2.2.1 and Section 2.2.4 where Witt vectors and syntomic cohomology are defined in the equivariant context. The second program is discussed in Section 2.2.2 and Section 2.2.3. In Section 2.2.2, I discuss an analogue of Hochschild homology equipped with a Pin(2)-action with potential applications to equivariant knot Floer homology. In Section 2.2.3, I discuss an approach to computations in C_p-equivariant homotopy theory using deformations, with potential applications to the odd primary Arf–Kervaire invariant problem.

2.2.1 Real topological Hochschild homology

Even though [HHR16] define norms for finite groups, it remains open to define norms more generally for compact Lie groups in general and it is under active investigation. So far it has only been done in certain cases. In [ABG+18a], they define norms $N_{\mu_n}^{\mathbb{T}}(R)$ from the groups of unity to the circle group $\mathbb{T}$. In fact, this is done by showing that the cyclic bar construction has the universal property of the norm. In joint work with Gerhardt and Hill [AKGH21], we prove an analogue of this result in the setting of Hochschild homology of rings with anti-involution, where the relevant bar construction is the dihedral bar construction

Theorem 2.1 ([AKGH21])**.** The dihedral bar construction satisfies the universal property of the norm $N_{C_2}^{O(2)}$.

One property of the norm for finite groups is the multiplicative double coset formula. For our new notion of norm $N_{D_2}^{O(2)}$, we also prove a such a formula, extending a result of [DMPP17] to dihedral groups D_{2m} of order $2m$ for $m > 1$.

Theorem 2.2 ([AKGH21])**.** There is an explicit muliplcative double coset formula for the norm $N_{D_2}^{O(2)}(R)$ restricted to D_{2m}.

We use these results to motivate a new algebraic definition of (algebraic) real Hochschild homology $\underline{\mathrm{HR}}_*^{D_{2m}}(\underline{M})$ of a ring with anti-involution $\underline{M}$. We prove that this construction is equipped with structure maps R, F and V. and use this to define the p-typical *real Witt vectors* $\underline{\mathbb{W}}(\underline{M};p)$. We then compute $\underline{\mathbb{W}}(\underline{\mathbb{Z}};p)$ as an example.

2.2.2 Homology of twisted G-rings

In joint work with Péroux and Merling [AKMP], I extend the theory of norms to certain more general compact Lie groups. To do this, I use the theory of crossed simplicial groups. The input of our construction is a twisted G-ring, defined as follows. Prerequistely, a group with parity is a group G with a group homomorphism $\varphi\colon G \to C_2 = \{-1, 1\}$. A twisted G-ring associated to a group with parity (G, φ) is then a ring with a G-action by abelian group homomorphisms where g acts by a ring homomorphism if $\varphi(g) = 1$ and a ring anti-homomorphism if $\varphi(g) = -1$.

Our construction also uses the theory of crossed simplicial groups developed by Z. Feidorowicz and J. L. Loday [FL91]. Crossed simplicial groups $\Delta\mathbf{G}$ are a common generalization of simplicial groups and Connes' cyclic category Λ, which plays a prominent role in the theory of Hochschild homology. Given a crossed simplicial group $\Delta\mathbf{G}$, we construct a notion of Hochschild homology TH$\mathbf{G}$ whose input is a twisted G_0-ring, where (G_0, φ) is a certain group with parity associated to the crossed simplicial group $\Delta\mathbf{G}$.

In the study of twisted G-rings, we prove the following result, which we state as a pre-theorem since it is work in progress.

Pre-theorem 2.1. The category $\Delta\mathbf{G}_+$, given adjoining an initial object to the crossed simplicial group $\Delta\mathbf{G}$, encodes the structure of a twisted G-ring in a precise sense.

Using this, we give a Nikolaus–Scholze-type construction of $\mathrm{TH}\mathbf{G}(R)$ providing new combinatorial models for norm functors for certain compact Lie groups.

Of particular interest is the case of the quaternioic category $\Delta\mathbf{Q}$ whose associated Hochschild homology gives a construction of the norm

$$N_{C_4}^{\mathrm{Pin}(2)}(R).$$

We aim to use $N_{C_4}^{\mathrm{Pin}(2)}(R)$ to study involutive knot Floer homology in analogy with the relationship between Hochschild homology and knot Floer homology.

Long-Term Goal 2.3. Construct a Hodge-to-de Rham spectral sequence for $N_{C_4}^{\mathrm{Pin}(2)}(R)$ that can be used to study equivariant knot Floer homology.

As an example computation, we compute $\mathrm{TH}\mathbf{G}(R[M])$ where R is a twisted G_0-ring and M is a twisted G_0-monoid.

2.2.3 A deformation of C_p-equivariant homotopy

The category of Artin–Tate $\mathbb{R}$-motivic homotopy theory can be regarded as a deformation of Sp^{C_2} by [BHS20]. In particular, there is a Beti-realization functor

$$\mathrm{Be}_{C_2}\colon \mathrm{SH}^{\mathrm{AT}}(\mathbb{R}) \longrightarrow \mathsf{Sp}^{C_2}$$

that can be used to simplifiy computations in C_2-equivariant homotopy. In the case of a cyclic group of order p where p is an odd prime, denoted C_p, there is not a known good replacement for $\mathrm{SH}(\mathbb{R})$. In work in progress with Behrens, Belmont, and Kong [AKBBK], we give an artificial construction of a category equipped with a realization functor to C_p-equivariant homotopy using the theory of deformations of stable homotopy theories.

Deformations of stable homotopy theories were developed in work of [GIKR18] and [Pst18]. They were used by Burkland–Hahn–Senger [BHS20] to extend the cellular $\mathbb{R}$-motivic stable homotopy category in a way that allows for realization functors recovering more information about the homotopy groups of C_2-spectra.

Pre-theorem 2.2. We construct a deformation of Borel C_p-spectra that recovers the a-complete Artin–Tate $\mathbb{R}$-motivic homotopy category at the prime $p = 2$.

In future work, we plan construct a deformation of the full C_p-equivariant stable homotopy theory. To do this, we must construct deformations of each of the terms in the fracture square

$$\begin{array}{ccc} \mathsf{Sp}^{C_p} & \longrightarrow & \mathsf{Sp}^{C_p}[a^{-1}] \\ \downarrow & & \downarrow \\ \left(\mathsf{Sp}^{C_p}\right)^{\wedge}_{a} & \longrightarrow & \left(\mathsf{Sp}^{C_p}\right)^{\wedge}_{a}[a^{-1}]. \end{array}$$

At $p = 2$, is possibly by work of Burkland–Hahn–Senger [BHS20] using C_2-equivariant cobordism, but at odd primes one does not have access to an analogue of the C_2-equivariant cobordism spectrum and therefore we take a different approach.

Goal 2.4. Construct a deformation of C_p-spectra that recovers the full Artin–Tate $\mathbb{R}$-motivic homotopy category at the prime $p = 2$.

Ravenel has suggested that a C_p-equivariant complex cobordism spectrum could be used to resolve the odd primary Arf–Kervaire invariant problem in analogy with the solution of the 2-primary Arf–Kervaire invariant problem in [HHR16]. We hope to use the deformation perspective to side-step the fact that such a C_p-equivariant complex cobordism spectrum is not known to exist.

2.2.4 Syntomic cohomology for rings with anti-involution

In a completely different vein, one may attempt to carry out a thorough study of the *arithmetic of equivariant ring spectra* and this the goal of my emmerging research program. For rings with anti-involution, there is not very much that is known in this context. For example C_2-equivariant prismatic cohomology and syntomic cohomology of rings have not been investigated. In joint work with J.D. Quigley [AKQ], we propose to use a strongly even version of the even filtration of [HRW22] to produce such a theory. We call this filtration on real topological Hochschild homology the real motivic filtration in analogy with the filtration of [HRW22] which recovers the more classical motivic filtration in certain cases.

Goal 2.5. Compute $\mathrm{THR}(BP_{\mathbb{R}}\langle 1\rangle)$ using the real motivic spectral sequence.

We also aim to understand syntomic/prismatic cohomology for suitable genuine commutative C_2-ring spectra

Goal 2.6. Develop the theory of C_2-equivariant chromatically quasisyntomic genuine commutative C_2-ring spectra and the notions of syntomic/prismatic cohomology.

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