

# MAPS OF SIMPLICIAL SPECTRA WHOSE REALIZATIONS ARE COFIBRATIONS.

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ABSTRACT. Given a map of simplicial topological spaces, mild conditions on degeneracies and the levelwise maps imply that the geometric realization of the simplicial map is a cofibration. These conditions are not formal consequences of model category theory, but depend on properties of spaces, and similar results have not been available for any model for the stable homotopy category. In this paper we prove such results for symmetric spectra. Consequently, we get a set of conditions which ensure that the geometric realization of a map of simplicial symmetric spectra is a cofibration. These conditions are very “user-friendly” in that they are simple, often easily checked, and do not require computation of a latching object or any other knowledge of Reedy theory.

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## 1. INTRODUCTION.

Let  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  be a morphism of simplicial objects in some category of topological interest (e.g. spaces, spectra, equivariant spectra, ...). It sometimes happens that one needs to know whether the induced map of geometric realizations

$$(1.0.1) \quad |f_\bullet| : |X_\bullet| \rightarrow |Y_\bullet|$$

is a cofibration<sup>1</sup>.

It is well-known that, given a model category  $\mathcal{C}$ , there exists a model structure (called the *Reedy model structure* due to its origin in C. Reedy’s thesis [8]) on the category of simplicial objects in  $\mathcal{C}$  such that, if  $f_\bullet$  is a Reedy cofibration, then the map 1.0.1 is a cofibration in the original model category  $\mathcal{C}$ . The map  $f_\bullet$  is a *Reedy cofibration* if, for each nonnegative integer  $n$ , the latching comparison map

$$v_n : X_n \prod_{L_n X_\bullet} L_n Y_\bullet \rightarrow Y_n$$

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<sup>1</sup>This problem arose for the authors in the process of constructing the “THH-May spectral sequence” of [1], but the problem is a very general one of basic interest. The paper [1] makes essential use of the results of this paper.

is a cofibration in  $\mathcal{C}$ . Here  $L_n$  is a certain colimit called the *latching object* construction, whose definition we recall in Definition 2.1.

From the definition of a Reedy cofibration one can see that, in practical situations, it is often very difficult to check that a given map  $f_\bullet$  is a Reedy cofibration. The purpose of this paper is to give a straightforward, practical, often easily-checked set of conditions on a simplicial map of spectra  $f_\bullet$  which ensure that it is a Reedy cofibration, and hence that the map 1.0.1 is a cofibration of spectra.

The main approach is to try to “import” some classical results from unstable homotopy theory into the setting of symmetric spectra. In pointed topological spaces, it is typically quite easy to check that a simplicial map  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is a Reedy cofibration:

- (1) if the degeneracy maps in  $X_\bullet$  and  $Y_\bullet$  are all closed cofibrations (i.e.,  $X_\bullet$  and  $Y_\bullet$  are “good” simplicial spaces, in the sense of [10]), then an easy application of Lillig’s cofibration union theorem [5] shows that  $X_\bullet$  and  $Y_\bullet$  are each Reedy cofibrant. (The key observation that makes this proof work is that the latching space  $L_n X_\bullet$  is simply the union of the images of the degeneracy maps in  $X_\bullet$ , and so the natural map  $L_n X_\bullet \rightarrow X_n$  is trivially seen to be a monomorphism, and using Lillig’s theorem, a cofibration.)
- (2) Then one can show that, if  $X_\bullet$  and  $Y_\bullet$  are each Reedy cofibrant and each map  $f_n : X_n \rightarrow Y_n$  is a closed cofibration, then  $f_\bullet$  is a Reedy cofibration.

So, in pointed topological spaces, one only needs the degeneracy maps in  $X_\bullet$  and  $Y_\bullet$  to be closed cofibrations and for each map  $f_n : X_n \rightarrow Y_n$  to be a closed cofibration in order for the map 1.0.1 to be a cofibration. See [10] and [7] for these facts from classical homotopy theory.

Now one wants to be able to do something similar in stable homotopy theory, i.e., to replace spaces with (some model for) spectra. Problems immediately arise: for example, it is no longer necessarily true that the latching object  $L_n X_\bullet$  is the “union of the images of the degeneracy maps” when  $X_\bullet$  is a simplicial spectrum, so the classical proofs of steps 1 and 2, above, do not “work” in spectra.

In this paper we devise proofs that *do* work. Throughout, we work in the setting of symmetric spectra, of [4]; see [9] for an excellent introduction and reference for symmetric spectra. Our symmetric spectra are allowed to take values in any “graded-concrete pointed model category”  $\mathcal{C}$  satisfying a mild technical condition described in Theorem 2.4. See Definition 3.1 for the (new) definition of a “graded-concrete pointed model category.” The purpose of this definition is that, letting  $\mathcal{C}$  be the category of simplicial sets, we recover the usual category of symmetric spectra in simplicial sets, which is a model for the stable homotopy category; but we can also let  $\mathcal{C}$  be other examples of graded-concrete pointed model categories, e.g.  $G$ -sets for a group  $G$ , so that our results apply to some non-classical stable homotopy theories as well. Our main results are as follows:

- In Theorem 3.4 we prove that, if  $X_\bullet$  is a simplicial symmetric spectrum satisfying the conditions:
  - the spectrum  $X_n$  is flat-cofibrant for each  $n$ ,
  - and each of the degeneracy maps  $s_i : X_n \rightarrow X_{n+1}$  is a levelwise cofibration,
then  $X_\bullet$  is Reedy flat-cofibrant.

If we furthermore assume that each spectrum  $X_n$  is *positive* flat-cofibrant, then  $X_n$  is Reedy positive flat-cofibrant.
- In Theorem 4.1 we prove that, if  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is a map of simplicial symmetric spectra satisfying the conditions:

– the map  $f_\bullet$  is a pointwise flat cofibration. That is, for each nonnegative integer  $n$ , the map  $f_n : X_n \rightarrow Y_n$  is a flat cofibration,  
 – and the simplicial spectrum  $Y_\bullet$  is Reedy levelwise-cofibrant,  
 then  $f_\bullet$  is a Reedy levelwise cofibration. Consequently  $|f_\bullet| : |X_\bullet| \rightarrow |Y_\bullet|$  is a levelwise cofibration.

If we furthermore assume that  $f_\bullet$  is a pointwise *positive* flat cofibration, then  $f_\bullet$  is a Reedy positive levelwise cofibration, and consequently  $|f_\bullet|$  is a positive levelwise cofibration.

• In Theorem 4.2 we prove that, if  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is a map of simplicial symmetric spectra satisfying the conditions:

- the map  $f_\bullet$  is a pointwise flat cofibration,
- and  $X_\bullet$  and  $Y_\bullet$  are both Reedy flat-cofibrant,

then  $f_\bullet$  is a Reedy flat cofibration. Consequently  $|f_\bullet| : |X_\bullet| \rightarrow |Y_\bullet|$  is a flat cofibration.

If we furthermore assume that  $f_\bullet$  is a pointwise *positive* flat cofibration, then  $f_\bullet$  is a Reedy positive flat cofibration, and consequently  $|f_\bullet|$  is a positive flat cofibration.

We make the distinction between “levelwise cofibrations,” “flat cofibrations,” “positive levelwise cofibrations,” and “positive flat cofibrations” because the category of symmetric spectra has more than one useful notion of cofibration: see Definition 2.3 for a review of their definitions. The brief version is that levelwise cofibrations have a simple definition which is often easy to verify for a given map, while flat cofibrations give rise to a better-behaved model category (symmetric spectra with flat cofibrations admit a *symmetric monoidal* stable model structure, while with levelwise cofibrations the model structure fails to be monoidal) but the defining condition for a map to be a flat cofibration is significantly more difficult to verify. Every flat cofibration is a levelwise cofibration, but the converse implication does not hold. Their positive versions require that one additional axiom be satisfied (see Definition 2.3), but have the advantage that  $E_\infty$ -ring spectra are more easily modelled in the positive flat model structure on symmetric spectra (of topological spaces or simplicial sets); see [4] or [9] for more details.

So, putting Theorems 3.4 and 4.2 together, we get a result which has some “teeth”:

**Theorem 1.1.** *Let  $C$  be any graded-concrete model category satisfying the assumptions of Theorem 2.4 (for example, let  $C$  be simplicial sets if one wants to wind up with a category of symmetric spectra which models the classical stable homotopy category). Let  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  be a map of simplicial symmetric spectra in  $C$ . Suppose that all of the following conditions are satisfied:*

- the symmetric spectra  $X_n$  and  $Y_n$  are flat-cofibrant for all  $n$ ,
- each of the degeneracy maps  $s_i : X_n \rightarrow X_{n+1}$  and  $s_i : Y_n \rightarrow Y_{n+1}$  are levelwise cofibrations,
- and  $f_n : X_n \rightarrow Y_n$  is a flat cofibration for all  $n$ .

Then  $f_\bullet$  is a Reedy flat cofibration. Consequently the map of symmetric spectra

$$|f_\bullet| : |X_\bullet| \rightarrow |Y_\bullet|$$

is a flat cofibration.

If we furthermore assume that each  $X_n$  and each  $Y_n$  is positive flat-cofibrant and that each  $f_n$  is a positive flat cofibration, then  $f_\bullet$  is a Reedy positive flat cofibration, and consequently the map of symmetric spectra  $|f_\bullet|$  is a positive flat cofibration.

It is a pleasure to thank E. Riehl for helping us with our questions when we were trying to find out whether the results in this paper were already known.

**Conventions 1.2.** Here are some running conventions which will be in force throughout this paper.

- In this paper, we consider symmetric spectra in a pointed simplicial model category  $\mathcal{C}$ , as in [9]. This category of symmetric spectra has several important notions of cofibration, and four of them are used in this paper: the *levelwise cofibrations*, the *flat cofibrations*, the *positive levelwise cofibrations*, and the *positive flat cofibrations*. As a consequence, we have four “Reedy” notions of cofibration in the category of simplicial symmetric spectra. To keep them distinct, we will speak of “Reedy levelwise cofibrations” as opposed to “Reedy flat cofibrations,” and “Reedy levelwise-cofibrant simplicial objects” as opposed to “Reedy flat-cofibrant simplicial objects,” and so on.
- When  $X$  is a symmetric spectrum in  $\mathcal{C}$  and  $n$  is a nonnegative integer, we will write  $X(n)$  for the level  $n$  object of  $X$ . When  $X_\bullet$  is a simplicial object, we will write  $X_n$  for the  $n$ -simplices object of  $X_\bullet$ . So, for example, given a simplicial symmetric spectrum  $X_\bullet$ , we write  $X(n)_\bullet$  for the simplicial object of  $\mathcal{C}$  whose  $m$ -simplices object is  $X(n)_m$ . The symbols  $X(n)_m$  and  $X_m(n)$  have the same meaning, and we will use them interchangeably.
- The word “levelwise” is used in two very different ways when speaking of maps between simplicial symmetric spectra, and for the sake of clarity, in this paper we consistently use the word “pointwise” instead of “levelwise” for one of these two notions. Here is a definition from [9]: given a map of symmetric spectra  $f : X \rightarrow Y$ , one says that  $f$  is a *levelwise cofibration* if each of the component maps  $f(n) : X(n) \rightarrow Y(n)$  is a cofibration in  $\mathcal{C}$ . (See Definition 2.3, below, for this definition and the related notion of a “flat cofibration.”)  
To distinguish this usage of the word “levelwise” from how the word “levelwise” is used when speaking of a map between simplicial objects, whenever we have a map  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  of *simplicial* symmetric spectra, we will say that  $f_\bullet$  is a *pointwise flat cofibration* if the map of symmetric spectra  $f_n : X_n \rightarrow Y_n$  is a flat cofibration for each  $n$ .
- When working with simplicial symmetric spectra, we will need to notationally distinguish between latching objects of simplicial objects, and latching objects of symmetric spectra; these notions are related but distinct (see Definition 2.1 and Remark 2.2). We will write  $\tilde{L}_n(X_\bullet)$  for the  $n$ th latching object of a simplicial object  $X_\bullet$ , and we write  $L_n(X)$  for the  $n$ th latching object of a symmetric spectrum  $X$  (see Construction I.5.29 of [9] for this second notion).

## 2. REVIEW OF THE RELEVANT MODEL STRUCTURES.

The definition of the latching object of a simplicial object dates back to Reedy’s thesis [8], but there are a number of different (equivalent) versions of that definition. The following version, which is convenient for what we do in this paper, appears as Remark VII.1.8 in [2].

**Definition 2.1.** Let  $X_\bullet$  be a simplicial object in a finitely complete and finitely cocomplete pointed category  $\mathcal{A}$ . Let  $\tilde{L}_0 X_\bullet$  be the zero object in  $\mathcal{A}$ . For  $n > 1$  define

$$\tilde{L}_n X_\bullet := \text{coeq} \left\{ \coprod_{0 \leq i < j \leq n-1} X_{n-2}\{i, j\} \begin{array}{c} \xrightarrow{S'} \\ \xrightarrow{S''} \end{array} \coprod_{k=0}^{n-1} X_{n-1}\{k\} \right\}$$

with  $S'$  and  $S''$  defined as follows: for a given pair  $(i, j)$  with  $i < j$  we define maps

$$X_{n-2}\{i, j\} \xrightarrow{s_i} X_{n-1}\{j\} \xrightarrow{\iota_j} \coprod_{k=0}^{n-1} X_{n-2}\{k\}$$

and

$$X_{n-2}\{i, j\} \xrightarrow{s_{j-1}} X_{n-1}\{i\} \xrightarrow{\iota_i} \coprod_{k=0}^{n-1} X_{n-2}\{k\}$$

where  $s_i, s_{j-1}$  are the degeneracy maps in our simplicial object and  $\iota_k$  is the inclusion into the  $k$ -th summand. We then define  $S'$  using the first collection of maps and the universal property of the coproduct and we define  $S''$  using the second collection of maps and the universal property of the coproduct. We have a map  $\coprod_{k=0}^{n-1} X_{n-1}\{k\} \rightarrow X_n$  given by the coproduct of the degeneracies and this produces a map  $\tilde{L}_n X_\bullet \rightarrow X_n$  by universal property of the coequalizer and the simplicial identity  $s_j s_i = s_i s_{j+1}$ .

The latching object  $\tilde{L}_n X_\bullet$  comes equipped with a natural comparison map  $\tilde{v}_n(X_\bullet) : \tilde{L}_n(X_\bullet) \rightarrow X_n$  which makes  $\tilde{v}_n$  a natural transformation of functors. (See [2] or [9] for more details on this comparison map.)

**Remark 2.2.** There also exists a version of latching objects for symmetric spectra. The definition of the latching objects  $L_n(X)$  of a symmetric spectrum  $X$  is lengthier and more involved than Definition 2.1, and as we do not need details of the construction in this paper, we do not reproduce it here; see Construction I.5.29 of [9] for the definition and some discussion. The essential property of the latching construction  $L_n$  that we will use in this paper is:

- $L_n(X)$  comes equipped with a natural comparison map  $v_n(X) : L_n(X) \rightarrow X(n)$  which makes  $L_n$  a natural transformation of functors.

The comparison map  $v_n$  plays an essential role in Definition 2.3.

Definition 2.3 is repeated almost verbatim from Definition III.2.1 and Construction I.5.29 of [9]:

**Definition 2.3.** Let  $C$  be a pointed simplicial model category and let  $\mathrm{Sp}_C$  denote the category of symmetric spectra in  $C$  (see section III.1 of [9]). A map  $f : X \rightarrow Y$  in  $\mathrm{Sp}_C$  is said to be:

- a levelwise cofibration if, for all nonnegative integers  $n$ , the map  $f(n) : X(n) \rightarrow Y(n)$  is a cofibration in  $C$ ,
- a positive levelwise cofibration if  $f$  is a levelwise cofibration and  $f(0) : X(0) \rightarrow Y(0)$  is an isomorphism,
- a flat cofibration if, for all nonnegative integers  $n$ , the latching map  $v_n f : X(n) \coprod_{L_n X} L_n Y \rightarrow Y(n)$  is a cofibration in  $C$ ,
- and a positive flat cofibration if  $f$  is a flat cofibration and  $f(0) : X(0) \rightarrow Y(0)$  is an isomorphism.

Theorem 2.4 is repeated almost verbatim from Theorem III.3.8 of [9]:

**Theorem 2.4.** Let  $C$  be a pointed simplicial model category such that, for every nonnegative integer  $n$ , the category of  $\Sigma_n$ -equivariant objects of  $C$  admits the mixed equivariant model structure. (See section III.3 of [9] for details; the motivating example here is when  $C$  is the category of simplicial sets.) Then the flat cofibrations are the cofibrations of a model structure on  $\mathrm{Sp}_C$  called the absolute flat stable model structure, while the positive flat cofibrations are the cofibrations of a model structure on  $\mathrm{Sp}_C$  called the positive flat stable model structure. Equipped with either of these model structures,  $\mathrm{Sp}_C$  is a symmetric monoidal stable model category satisfying the pushout-product axiom, and when  $C$  is the

category of pointed simplicial sets, the homotopy category  $\mathrm{Ho}(\mathrm{Sp}_C)$  is equivalent to the classical stable homotopy category, with its smash product.

The category of symmetric spectra does *not* have a well-behaved symmetric monoidal model category when equipped with the levelwise cofibrations or the positive levelwise cofibrations. See [9] for details.

The absolute flat model structure and the positive flat model structure have equivalent homotopy categories, and have mostly exactly the same good properties. The absolute flat model structure has the advantage of being slightly simpler, while the positive flat model structure has some better properties than the absolute flat model structure when one wants to work with structured symmetric ring spectra. Again, see [9] for details.

### 3. REEDY COFIBRANT OBJECTS: SUFFICIENT CONDITIONS.

**Definition 3.1.** Let  $\mathrm{Sets}_*$  denote the category of pointed sets, and let  $\mathbb{N}$  denote the category with one object for each nonnegative integer, and with no non-identity morphisms. By a graded pointed set we mean a functor  $F : \mathbb{N} \rightarrow \mathrm{Sets}_*$ , and by the category of graded sets we mean the category  $\mathrm{Sets}_*^{\mathbb{N}}$  of functors from  $\mathbb{N}$  to  $\mathrm{Sets}_*$ . We will refer to  $F(n)$  as the degree  $n$  set of  $F$ .

By a graded-concrete pointed model category we mean a pointed model category  $C$  such that there exists a functor  $U : C \rightarrow \mathrm{Sets}_*^{\mathbb{N}}$  satisfying the following axioms:

- $U$  preserves finite limits and finite colimits, and
- a map  $f : X \rightarrow Y$  in  $C$  is a cofibration if and only if  $Uf$  is a monomorphism.

It is very easy to show that the monomorphisms in  $\mathrm{Sets}_*^{\mathbb{N}}$  are the natural transformations  $F \rightarrow G$  such that the map of pointed sets  $F(n) \rightarrow G(n)$  is one-to-one for all  $n \in \mathbb{N}$ .

**Example 3.2.** The motivating example of a graded-concrete pointed model category is the category  $\mathrm{Sets}_*^{\Delta^{\mathrm{op}}}$  of simplicial pointed sets, with the usual Kan model structure: to get a functor  $U : \mathrm{Sets}_*^{\Delta^{\mathrm{op}}} \rightarrow \mathrm{Sets}_*^{\mathbb{N}}$  as in Definition 3.1, we can let  $(U(X_\bullet))(n) = X_n$ , i.e.,  $U$  is the functor that simply forgets the faces and degeneracies in a simplicial pointed set. As the category of simplicial pointed sets is a functor category, limits and colimits in simplicial pointed sets are computed pointwise [6]. Consequently  $U$  preserves limits and colimits, and since the cofibrations in the Kan model structure on simplicial pointed sets are simply the monomorphisms (which are exactly the pointwise monomorphisms, in the sense of Conventions 1.2),  $f$  is a cofibration if and only if  $Uf$  is a monomorphism.

It is certainly not the case that every pointed model category is graded-concrete. For example, one sees easily from Definition 3.1 that, if  $g \circ f$  is a cofibration in a graded-concrete model category, then  $f$  is also a cofibration, i.e., cofibrations are “retractile.” There are many model categories which do not satisfy this condition.

Now we will say what we mean by a “good” simplicial symmetric spectrum. In classical references on simplicial spaces (see e.g. [10]), a simplicial topological space is called “good” if its degeneracy maps are all closed cofibrations; our definition is modelled on this classical definition.

**Definition 3.3.** Let  $C$  be a graded-concrete pointed model category satisfying the assumptions of Theorem 2.4. By a good simplicial symmetric spectrum in  $C$  we mean a simplicial object  $X_\bullet$  in the category  $\mathrm{Sp}_C$  such that

- (1)  $X_n$  is a flat-cofibrant symmetric spectrum for each  $n$ , and
- (2) the degeneracy maps  $s_i : X_n \rightarrow X_{n+1}$  are levelwise cofibrations for each  $n$  and  $i$ .

We will say that  $X_\bullet$  is positive-good if  $X_\bullet$  is good and  $X_n$  is positive flat-cofibrant for each  $n$ .

Definition 3.3 is unusual-looking, because it refers to two different model structures (“flat” and “levelwise”). Here is some explanation: every flat cofibration is also a levelwise cofibration, so if  $X_\bullet$  is a simplicial symmetric spectrum which is pointwise flat-cofibrant and whose degeneracies are all flat cofibrations, then  $X_\bullet$  is good. In Definition 3.3, however, we only ask for the degeneracies to be levelwise cofibrations, not flat cofibrations, because:

- (1) checking that a map is a levelwise cofibration in a specific case of interest is typically much easier than checking that it is a flat cofibration, and
- (2) our main theorem in this section, Theorem 3.4, only needs the degeneracy maps to be levelwise cofibrations, not necessarily flat cofibrations.

Finally, we state the main theorem in this section of the paper, which provides a typically easily-checked criterion for a simplicial symmetric spectrum to be Reedy flat-cofibrant. The rest of this section is devoted to proving this theorem.

**Theorem 3.4.** *Let  $\mathcal{C}$  be a graded-concrete pointed model category satisfying the assumptions of Theorem 2.4. Let  $X_\bullet$  be a good simplicial symmetric spectrum in  $\mathcal{C}$ . Then  $X_\bullet$  is Reedy flat-cofibrant; i.e. the map  $\tilde{L}_n X_\bullet \rightarrow X_n$  is a flat cofibration for each  $n$ .*

*If  $X_\bullet$  is furthermore assumed to be positive-good, then  $X_\bullet$  is Reedy positive flat-cofibrant.*

We will break the proof into several short steps. First, we recall a result of Hirschhorn that will be useful for our purposes. There are multiple Reedy model structures that one could put on the category of bisimplicial objects in a model category, by considering a bisimplicial object as a simplicial object in the category of simplicial objects either “vertically” or “horizontally,” and by considering a bisimplicial object as a functor out of the Reedy category  $\Delta^{\text{op}} \times \Delta^{\text{op}}$ .

We will refer to each of these as “the Reedy pointwise-model structure” and not distinguish them, because the following theorem of Hirschhorn states that they are all the same.

**Theorem 3.5** (Hirschhorn’s Theorem 15.5.2 [3]). *Let  $\mathcal{C}$  and  $\mathcal{D}$  be Reedy categories and let  $\mathcal{M}$  be a model category. Then the category  $\mathcal{M}^{\mathcal{C} \times \mathcal{D}}$  has the same Reedy model structure when viewed as:*

- (1) diagrams in  $\mathcal{M}$  indexed by the Reedy category  $\mathcal{C} \times \mathcal{D}$ ,
- (2) the category  $(\mathcal{M}^{\mathcal{C}})^{\mathcal{D}}$ ; i.e. diagrams in  $\mathcal{M}^{\mathcal{C}}$  indexed by the Reedy category  $\mathcal{D}$ , or
- (3) the category  $(\mathcal{M}^{\mathcal{D}})^{\mathcal{C}}$ ; i.e. diagrams in  $\mathcal{M}^{\mathcal{D}}$  indexed by the Reedy category  $\mathcal{C}$ .

We now begin the propositions and lemmas that lead to a proof of Theorem 3.4.

**Lemma 3.6.** *Equip the category  $\text{Sets}_*$  of pointed sets with the model structure in which the cofibrations are the injections, the fibrations are the surjections, and all maps are weak equivalences. Then every simplicial object of  $\text{Sets}_*$  is Reedy-cofibrant.*

*Proof.* This is not difficult to deduce from Corollary 15.8.8—that is, all bisimplicial sets are Reedy cofibrant—from Hirschhorn’s book [3]. A direct proof is even easier: given a simplicial pointed set  $X_\bullet$ , the latching comparison map  $v_n : L_n X_\bullet \rightarrow X_n$  is an isomorphism onto the collection of degenerate  $n$ -simplices in  $X_\bullet$ . Consequently  $v_n$  is a monomorphism, consequently  $X_\bullet$  is Reedy cofibrant.  $\square$

**Observation 3.7.** Let  $\mathbb{N}$  be as in Definition 3.1, i.e.,  $\mathbb{N}$  is the category with one object for each natural number, and no non-identity morphisms. Then  $\mathbb{N}$  has a unique Reedy structure, and for any model category  $\mathcal{C}$ , an object  $F$  in the functor category  $\mathcal{C}^{\mathbb{N}}$  is Reedy cofibrant if and only if  $F(n)$  is cofibrant (in  $\mathcal{C}$ ) for each  $n \in \mathbb{N}$ . (We use this observation in the proof of Proposition 3.8.)

Now we show that a simplicial symmetric spectrum in a graded-concrete pointed model category is always Reedy levelwise-cofibrant.

**Proposition 3.8.** *Let  $\mathcal{C}$  be a graded-concrete pointed model category satisfying the assumptions of Theorem 2.4. Let  $X_\bullet$  be an object in the category  $\mathrm{Sp}_C^{\Delta^{\mathrm{op}}}$  of simplicial symmetric spectra in  $\mathcal{C}$ . Then  $X_\bullet$  is Reedy levelwise-cofibrant; i.e. the map  $\tilde{L}_n X_\bullet \rightarrow X_n$  of symmetric spectra is a levelwise cofibration for each  $n$ .*

*Proof.* Choose a functor  $U : \mathcal{C} \rightarrow \mathrm{Sets}_*^{\mathbb{N}}$  as in Definition 3.1. We first consider, for any fixed  $n$ , the simplicial object  $X(n)_\bullet$  of  $\mathcal{C}$  obtained by taking the level  $n$  object of each of the symmetric spectra in  $X_\bullet$ . Applying  $U$  to this simplicial object yields a simplicial graded pointed set, i.e., a functor  $H : \Delta^{\mathrm{op}} \rightarrow \mathrm{Sets}_*^{\mathbb{N}}$ , equivalently, a functor  $H' : \mathbb{N} \rightarrow \mathrm{Sets}_*^{\Delta^{\mathrm{op}}}$ . In the Reedy model structure on  $\mathrm{Sets}_*^{\Delta^{\mathrm{op}}}$  (obtained by equipping  $\mathrm{Sets}_*$  with the model structure in which the cofibrations are the monomorphisms), every object is cofibrant, by Lemma 3.6. Observation 3.7 tells us that  $H'$  is cofibrant in the Reedy model structure on  $(\mathrm{Sets}_*^{\Delta^{\mathrm{op}}})^{\mathbb{N}}$ , consequently (using Theorem 3.5) cofibrant in  $\mathrm{Sets}_*^{\Delta^{\mathrm{op}} \times \mathbb{N}}$ , consequently (again using Theorem 3.5)  $H$  is Reedy cofibrant in  $(\mathrm{Sets}_*^{\mathbb{N}})^{\Delta^{\mathrm{op}}}$ . So the latching comparison map

$$\tilde{L}_m(UX(n)_\bullet) \rightarrow (UX(n))_m$$

is a monomorphism of graded pointed sets. Since  $U$  preserves finite limits and finite colimits, it commutes with the construction of latching objects; hence  $U(\tilde{L}_m(X(n)_\bullet)) \rightarrow U(X(n)_m)$  is a monomorphism of graded pointed sets, hence (by Definition 3.1) the latching comparison map

$$(3.0.1) \quad \tilde{L}_m(X(n)_\bullet) \rightarrow X(n)_m$$

is a cofibration in  $\mathcal{C}$  for each  $m$  and each  $n$ .

Now we wish to show that the map of symmetric spectra

$$(3.0.2) \quad \tilde{L}_m X_\bullet \rightarrow X_m$$

is a levelwise cofibration. By Definition 2.1,

$$\tilde{L}_m X_\bullet := \mathrm{coeq} \left\{ \coprod_{0 \leq i < j \leq k} X_{m-2}\{i, j\} \rightrightarrows \coprod_{k=0}^{m-1} X_{m-1}\{k\} \right\}.$$

Since colimits are computed levelwise in the category of symmetric spectra (see Example I.3.5 [9])

$$(\tilde{L}_m X_\bullet)(n) \cong \tilde{L}_m(X(n)_\bullet).$$

Now, since the map 3.0.1 is a cofibration, we know that the map

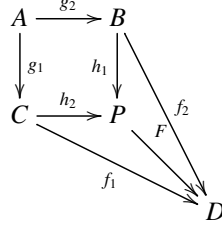
$$(\tilde{L}_m X_\bullet)(n) \rightarrow (X_m)(n)$$

is a cofibration of simplicial sets for each  $n$ , and therefore the map 3.0.2 is a levelwise cofibration. This is exactly the condition, from Definition 2.3 and the definition of Reedy cofibrations, for  $X_\bullet$  to be Reedy levelwise-cofibrant.  $\square$

The following lemma is an easier version of Lillig's union theorem, from [5], and is likely well known.



**Lemma 3.9** (Lillig’s theorem for graded pointed sets). *Suppose we have a diagram in graded pointed sets of the form*

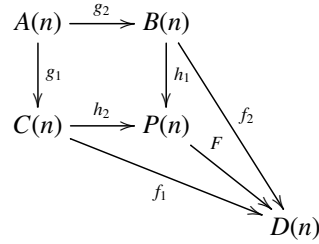


where  $P$  is the pushout of the upper left horn, and  $g_1, g_2, f_1$  and  $f_2$  are monomorphisms of graded pointed sets. Assume

$$A(n) = C(n) \cap B(n) \hookrightarrow D(n)$$

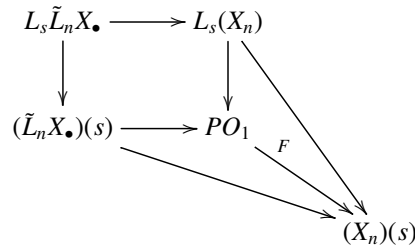
is an inclusion of sets for each  $n$ . Then  $F : P \rightarrow D$  is a monomorphism.

*Proof.* Colimits and limits are computed pointwise in graded pointed sets because it is a category of functors (see e.g. [6]). For any  $n$  there is a commutative diagram in the category of sets



where  $g_1, g_2, f_1$  and  $f_2$  are monomorphisms. The fact that  $f_1$  and  $f_2$  are monomorphisms implies that  $h_1$  and  $h_2$  are monomorphisms, since monomorphisms are “retractile,” i.e.,  $g \circ f$  is monic implies that  $f$  is monic. Let  $x, y \in P(n)$  such that  $F(x) = F(y)$ . We break into cases and use the fact that the pushout  $P(n)$  is simply the union  $C(n) \coprod_{A(n)} B(n)$ . If  $x, y \in B(n) \subset P(n)$  or if  $x, y \in C(n) \subset P(n)$ , then since  $f_1$  and  $f_2$  are monomorphisms, then  $x = y$ . If  $x \in B(n) \subset P(n)$  and  $y \in C(n) \subset P(n)$ , then  $F(x) = F(y)$  implies that  $x, y \in B(n) \cap C(n) \subset D(n)$ . If  $x, y \in B(n) \cap C(n)$  then since  $A(n) = C(n) \cap B(n) \hookrightarrow D(n)$  is a monomorphism of sets, we know  $x = y$ .  $\square$

In order to prove that  $X_\bullet \in \text{ob Sp}_C^{\text{Aop}}$  is Reedy flat-cofibrant and not just Reedy levelwise-cofibrant, we need to know that the map  $F$  in the commutative diagram



is a cofibration in  $\mathcal{C}$ , where  $PO_1$  is defined as the pushout of the upper left horn.

**Proposition 3.10.** *Let  $\mathcal{C}$  be a graded-concrete pointed model category satisfying the assumptions of Theorem 2.4, and let  $X_\bullet$  be a pointwise flat-cofibrant simplicial symmetric spectrum in  $\mathcal{C}$ . Define  $PB$  to be the pullback of the lower right horn in the diagram*

$$(3.0.3) \quad \begin{array}{ccc} PB & \longrightarrow & L_s(X_n) \\ \downarrow & & \downarrow \\ (\tilde{L}_n X_\bullet)(s) & \longrightarrow & (X_n)(s) \end{array}$$

in  $\mathcal{C}$ . Then the universal map  $F'$  in the commutative diagram

$$(3.0.4) \quad \begin{array}{ccccc} PB & \xrightarrow{g_2} & L_s(X_n) & & \\ \downarrow g_1 & & \downarrow & \searrow f_1 & \\ (\tilde{L}_n X_\bullet)(s) & \longrightarrow & PO_2 & \xrightarrow{F'} & (X_n)(s) \\ & \searrow f_2 & & \nearrow & \end{array}$$

in  $\mathcal{C}$  is a cofibration, where  $PO_2$  is the pushout of the upper left horn.

*Proof.* Choose a functor  $U : \mathcal{C} \rightarrow \mathbf{Sets}_*^{\mathbb{N}}$  as in Definition 3.1. Since limits and colimits in simplicial symmetric spectra are computed pointwise, and since  $U$  is assumed to preserve finite limits and finite colimits, the graded pointed set  $U(PB)$  is the pullback of the diagram

$$\begin{array}{ccc} & & UL_s(X_n) \\ & & \downarrow \\ U\tilde{L}_n(X_\bullet)(s) & \longrightarrow & U(X_n)(s), \end{array}$$

and the graded pointed set  $U(PO_2)$  is the pushout of the diagram

$$\begin{array}{ccc} U(PB) & \longrightarrow & UL_s(X_n) \\ \downarrow & & \\ U(\tilde{L}_n X_\bullet)(s) & & \end{array}$$

The map  $f_1 : L_s(X_n) \rightarrow (X_n)(s)$  in diagram 3.0.4 is a cofibration in  $\mathcal{C}$ , since the spectrum  $X_n$  is flat-cofibrant for each  $n$ , and the map  $f_2$  in diagram 3.0.4 is a cofibration in  $\mathcal{C}$  by Proposition 3.8. So  $Uf_1$  and  $Uf_2$  are monomorphisms of graded pointed sets, so the maps  $Ug_1$  and  $Ug_2$  are monic since they are pullbacks of monomorphisms.

Therefore, all the conditions of Lemma 3.9 are satisfied for  $U(X_\bullet)$ , i.e.,  $UF' : U(PO_2) \rightarrow U(X_n(s))$  is a monomorphism of graded pointed sets. So  $F'$  is a cofibration in  $\mathcal{C}$ , by Definition 3.1.  $\square$

Note that by universal properties, we have maps

$$(3.0.5) \quad G : L_s \tilde{L}_n X_\bullet \longrightarrow PB$$

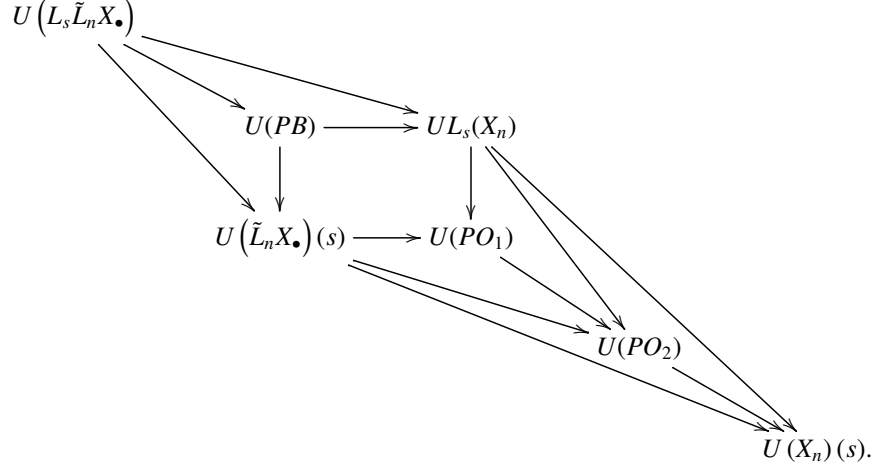
and

$$F'' : PO_1 \longrightarrow PO_2,$$

and each of these maps fits into Diagram 3.0.8, below. Since  $F'$  is a cofibration in  $\mathcal{C}$  and  $F = F' \circ F''$ , we just need to show that  $F''$  is a cofibration in  $\mathcal{C}$  and that will imply  $F$  is a cofibration in  $\mathcal{C}$ . We claim that we just need to prove that the map  $U(G)$  is a surjection,

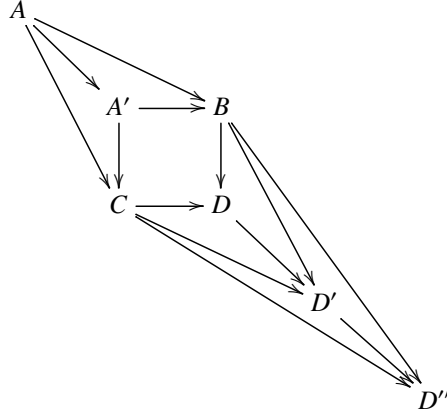
where  $U$  is as in Definition 3.1, and that will imply that  $F''$  is a cofibration as desired. The following lemma validates that claim, by choosing the sets in the statement of Lemma 3.11 so that diagram 3.0.7 is the diagram

(3.0.6)



**Lemma 3.11.** *Suppose we have a commutative diagram of pointed sets*

(3.0.7)



where the maps  $C \rightarrow D'$ ,  $B \rightarrow D'$  and  $D' \rightarrow D''$  are monomorphisms and  $A \rightarrow A'$  is an epimorphism. Suppose that  $D = B \coprod_A C$ ,  $D' = B \coprod_{A'} C$ , and  $A' = B \cap_{D'} C$ . Then the map  $D \rightarrow D'$  is a monomorphism.

*Proof.* Write  $H : D \rightarrow D'$  for the map in diagram 3.0.7. To show that  $H$  is a monomorphism, we pick  $x, y \in D$  such that  $H(x) = H(y)$ . We need to show that  $x = y$ . Since  $D = B \coprod_A C$ , either  $x, y \in B$ ,  $x, y \in C$ , or  $x \in B$  and  $y \in C$  such that both elements pull back to the same element in  $B \cap_D C$ . Suppose  $x, y \in B$ . Since  $B \rightarrow D'$  is a monomorphism, we know  $B \rightarrow D'$  is a monomorphism and this implies that  $x = y$ . Similarly, let  $x, y \in C$ . Then  $C \rightarrow D'$  is a monomorphism and hence  $C \rightarrow D'$  is a monomorphism, which implies  $x = y$ . If  $x \in B$  and  $y \in C$ , then both elements pull back to the same element  $\bar{u} \in A'$ . Since  $A \rightarrow A'$  is an epimorphism, there is an element  $u \in A$  that maps to  $\bar{u}$  in  $A'$ . The element  $u$  then must map to the element  $x = y \in D$  by commutativity of the diagram and the fact that  $D = B \coprod_A C$ .  $\square$

In order to apply Lemma 3.11 to diagram 3.0.6, we need to verify that the map  $U(G)$  (where  $G$  is as in 3.0.5) is an epimorphism. We verify this in the following proposition:

**Proposition 3.12.** *Let  $X_\bullet$  be pointwise flat-cofibrant, and let  $G$  be as in the commutative diagram*

$$(3.0.8) \quad \begin{array}{ccccc} L_s \tilde{L}_n X_\bullet & & & & \\ & \searrow^{g_2} & & & \\ & & PB & \xrightarrow{\ell_1} & L_s(X_n) \\ & \searrow^{G} & \downarrow \ell_2 & & \downarrow f_1 \\ & & (\tilde{L}_n X_\bullet)(s) & \xrightarrow{f_2} & PO_1 \\ & \searrow^{g_1} & & & \downarrow F'' \\ & & & & PO_2 \\ & & & & \downarrow F' \\ & & & & (X_n)(s) \end{array}$$

(Note: Additional arrows in the diagram include  $h_1$  from  $L_s(X_n)$  to  $PO_2$ ,  $h_2$  from  $(\tilde{L}_n X_\bullet)(s)$  to  $PO_2$ , and  $h_1$  from  $L_s(X_n)$  to  $(X_n)(s)$ , and  $h_2$  from  $(\tilde{L}_n X_\bullet)(s)$  to  $(X_n)(s)$ .)

Then the map of graded pointed sets  $U(G) : U(L_s \tilde{L}_n X_\bullet) \rightarrow U(PB)$  is an epimorphism.

*Proof.* Throughout, we use the assumption (from Definition 3.1) that  $U$  preserves finite limits and finite colimits. Let  $\bar{z} \in U(PB)$ . Then since  $U(PB)$  is a pullback,  $\bar{z}$  is represented by elements

$$x_1 = \ell_1(\bar{z}) \in U(L_s X_n) \text{ and } x_2 = \ell_2(\bar{z}) \in U((\tilde{L}_n X_\bullet)(s))$$

such that  $Uh_1(x_1) = Uh_2(x_2)$ . Since  $x_2 \in U(\tilde{L}_n X_\bullet(s)) = \coprod_{k=0}^{n-1} U(X_{n-1}(s))\{k\} / \sim$  it can be chosen as an equivalence class of some element in  $U(X_{n-1}(s))\{j\}$  for some  $j$ . Then we can apply the  $j$ -th face map, to produce the composite

$$X_n \xrightarrow{d_j} X_{n-1}\{j\} \longrightarrow \coprod_{k=0}^{n-1} X_{n-1}\{k\} \longrightarrow \coprod_{k=0}^{n-1} X_{n-1}\{k\} / \sim,$$

which we call  $\bar{d}_j$ . We write  $\bar{d}_j(s)$  for the map

$$X_n(s) \xrightarrow{d_j} X_{n-1}(s)\{j\} \longrightarrow \coprod_{k=0}^{n-1} X_{n-1}(s)\{k\} \longrightarrow \coprod_{k=0}^{n-1} X_{n-1}(s)\{k\} / \sim$$

and we write  $U(\bar{d}_j(s))$  for the map of graded pointed sets produced by applying the functor  $U$ . We chose  $j$  so that the function  $U(\bar{d}_j(s))$  satisfies the formula

$$U(\bar{d}_j(s) \circ h_2)(x_2) = x_2.$$

By functoriality of  $U$  and  $L_s$  there is also a map  $UL_s(\bar{d}_j) : UL_s X_n \rightarrow UL_s \tilde{L}_n X_\bullet$ . We claim the following:

- (1)  $U(g_1 \circ L_s(\bar{d}_j))(x_1) = x_2$ , and
- (2)  $U(g_2 \circ L_s(\bar{d}_j))(x_1) = x_1$ .

Item 1 follows by naturality of  $\nu_s$ , which we explain as follows. First, we know  $Ug_1 = U\nu_s(\tilde{L}_n X_\bullet)$  and naturality states that the diagram

$$\begin{array}{ccc} UL_s X_n & \xrightarrow{UL_s(\bar{d}_j)} & UL_s \tilde{L}_n X_\bullet \\ U\nu_s(X_n) \downarrow & & \downarrow U\nu_s(\tilde{L}_n X_\bullet) \\ U(X_n(s)) & \xrightarrow{U\bar{d}_j(s)} & U\tilde{L}_n(X_\bullet(s)) \end{array}$$

commutes; i.e.,

$$U(\nu_s(\tilde{L}_n X_\bullet) \circ L_s(\bar{d}_j))(x_1) = U(\bar{d}_j(s) \circ \nu_s(X_n))(x_1)$$

We then use the fact that  $\nu_s(X_n) = h_1$  and the formula  $Uh_1(x_1) = Uh_2(x_2)$  to produce

$$U(\bar{d}_j \circ \nu_s(X_n))(x_1) = U(\bar{d}_j(s))(U(h_1(x_1))) = U(\bar{d}_j)(Uh_2(x_2)).$$

This combines with the fact that  $U(\bar{d}_j \circ h_2)(x_2) = x_2$  to produce

$$U(g_1 \circ L_s(\bar{d}_j))(x_1) = x_2$$

as desired.

To prove Item 2, note that by naturality of  $\nu_s$  the diagram

$$\begin{array}{ccccc} UL_s(X_n) & \xrightarrow{UL_s(\bar{d}_j)} & UL_s \tilde{L}_n X_\bullet & \xrightarrow{UL_s(\tilde{\nu}(X_\bullet))} & UL_s(X_n) \\ U\nu_s(X_n) \downarrow & & \downarrow U\nu_s(\tilde{L}_n(X_\bullet)) & & \downarrow U\nu_s(X_n) \\ U(X_n(s)) & \xrightarrow{U\bar{d}_j(s)} & U\tilde{L}_n X_\bullet(s) & \xrightarrow{U\tilde{\nu}(X_\bullet)(s)} & U(X_n(s)) \end{array}$$

commutes. We know that  $Uh_1 = U\nu_s(X_n)$ , so in formulas

$$U(h_1 \circ L_s(\tilde{\nu}(UX_\bullet)) \circ L_s(\bar{d}_j))(x_1) = (\tilde{\nu}(UX_\bullet)(s)) \circ U\bar{d}_j(s) \circ Uh_1(x_1)$$

and since  $Uh_1(x_1) = Uh_2(x_2)$  and  $U(\bar{d}_j(s) \circ h_2)(x_2) = x_2$ , we know that

$$U(\bar{d}_j(s) \circ h_1)(x_1) = U(\bar{d}_j(s) \circ h_2)(x_2) = x_2,$$

and hence that

$$U(\tilde{\nu}(X_\bullet)(s) \circ \bar{d}_j \circ h_1)(x_1) = U(\tilde{\nu}(X_\bullet)(s) \circ \bar{d}_j(s) \circ h_2)(x_2) = U\tilde{\nu}(X_\bullet)(s)(x_2).$$

Now note that  $U\tilde{\nu}(X_\bullet)(s) = Uh_2$  so

$$U\tilde{\nu}(X_\bullet)(s)(x_2) = Uh_2(x_2) = Uh_1(x_1)$$

and hence

$$U(h_1 \circ L_s(\tilde{\nu}(X_\bullet)) \circ L_s(\bar{d}_j))(x_1) = Uh_1(x_1)$$

Since  $X_\bullet$  is pointwise flat-cofibrant,  $Uh_1$  is a monomorphism, so it is left cancellable and therefore

$$U(L_s(\tilde{\nu}(X_\bullet)) \circ L_s(\bar{d}_j))(x_1) = x_1.$$

Now note that  $Ug_2 = UL_s(\tilde{\nu}(X_\bullet))$  by definition, so we have proven the claim.

Thus, given an element  $\bar{z}$  in the pullback represented by  $x_1$  and  $x_2$ , we have constructed an element  $z = UL_s(\bar{d}_j)(x_1)$ , such that  $UG(z) = \bar{z}$  as desired.  $\square$

*Proof of Theorem 3.4.* The goal is to prove that a good symmetric spectrum in the flat model structure is Reedy flat-cofibrant. By Proposition 3.8, we know that a good symmetric spectrum in the flat model structure is Reedy levelwise-cofibrant, so we just need to

elevate the map  $\tilde{L}_n X_\bullet \rightarrow X_n$  to a flat cofibration; i.e., we want to know that the map  $F$  in the commutative diagram

$$\begin{array}{ccc}
 L_s \tilde{L}_n X_\bullet & \longrightarrow & L_s X_n \\
 \downarrow & & \downarrow \\
 \tilde{L}_n(X_\bullet(s)) & \longrightarrow & PO_1 \\
 & \searrow & \downarrow \\
 & & X_n(s)
 \end{array}$$

$F$

is a cofibration in  $\mathcal{C}$ , where  $PO_1$  is defined to be the pushout of the upper left horn. We then write  $F$  as a composite  $F' \circ F''$ . The map  $F'$  is a cofibration in  $\mathcal{C}$  by Proposition 3.10, and the map  $F''$  is a cofibration in  $\mathcal{C}$  by Lemma 3.11 and Proposition 3.12. So  $F$  is a cofibration.

Now suppose furthermore that  $X_\bullet$  is positive-good. We need to know that  $X_\bullet$  is also cofibrant in the Reedy positive-flat model structure, i.e., that  $v_n(X_\bullet) : \tilde{L}_n X_\bullet \rightarrow X_n$  is a positive flat cofibration. Since we have already shown that  $v_n$  is a flat cofibration, all that remains is to show that  $v_n(X_\bullet)(0) : (\tilde{L}_n X_\bullet)(0) \rightarrow X_n(0)$  is an isomorphism, i.e., that  $v_n(X(0)_\bullet) : \tilde{L}_n(X(0)_\bullet) \rightarrow X(0)_n$  is an isomorphism. Since  $X_\bullet$  is positive-good, each  $X_n$  is positive flat-cofibrant, so  $X(0)_n \cong 0$  for all  $n$ . So  $X(0)_\bullet$  is the zero simplicial object, so its latching objects are also all zero, so  $v_n(X(0)_\bullet)$  is an isomorphism, as desired.  $\square$

#### 4. REEDY COFIBRATIONS: SUFFICIENT CONDITIONS.

Recall that the definition of “graded-concrete pointed model category” was given in Definition 3.1.

**Theorem 4.1.** *Let  $\mathcal{C}$  be a graded-concrete pointed model category satisfying the assumptions of Theorem 2.4, and let  $X_\bullet \xrightarrow{f_\bullet} Y_\bullet$  be a morphism of simplicial symmetric spectra in  $\mathcal{C}$ . Make the following assumptions:*

- *The map  $f_\bullet$  is a pointwise flat cofibration. That is, for each nonnegative integer  $n$ , the map of spectra  $X_n \xrightarrow{f_n} Y_n$  is a flat cofibration.*
- *The simplicial spectrum  $Y_\bullet$  is Reedy levelwise-cofibrant.*

*Then  $f_\bullet$  is a Reedy levelwise cofibration.*

*If we furthermore assume that  $f_\bullet$  is a pointwise positive flat cofibration, then  $f_\bullet$  is also a Reedy positive levelwise cofibration.*

*Proof.* For each nonnegative integer  $n$ , we need to show that the map of symmetric spectra

$$X_n \coprod_{\tilde{L}_n(X_\bullet)} \tilde{L}_n(Y_\bullet) \rightarrow Y_n$$

is a levelwise cofibration, i.e., that

$$(4.0.1) \quad \left( X_n \coprod_{\tilde{L}_n(X_\bullet)} \tilde{L}_n(Y_\bullet) \right) (m) \rightarrow Y_n(m)$$

is a cofibration in  $\mathcal{C}$  for all nonnegative integers  $m, n$ . Now colimits in  $\mathrm{Sp}_{\mathcal{C}}$  are computed levelwise (see example I.3.5 in [9]), so map 4.0.1 agrees, up to an isomorphism, with the

map

$$(4.0.2) \quad c_n(m) : X_n(m) \coprod_{(\tilde{L}_n(X_\bullet))(m)} (\tilde{L}_n(Y_\bullet))(m) \rightarrow Y_n(m).$$

Since  $\mathcal{C}$  is assumed graded-concrete, choose a functor  $U : \mathcal{C} \rightarrow \mathbf{Sets}_*^{\mathbb{N}}$  as in Definition 3.1. If we can show that  $U(c_n(m))$  is a monomorphism, then  $c_n(m)$  is a cofibration, and we are done. Since  $U$  preserves finite colimits, the map  $U(c_n(m))$  agrees (up to an isomorphism in the domain) with the map

$$\tilde{c}_n(m) : U(X_n(m)) \coprod_{U((\tilde{L}_n(X_\bullet))(m))} U((\tilde{L}_n(Y_\bullet))(m)) \rightarrow U(Y_n(m)).$$

As in all functor categories, colimits in  $\mathbf{Sets}_*^{\mathbb{N}}$  are computed pointwise and the monomorphisms in  $\mathbf{Sets}_*^{\mathbb{N}}$  are the pointwise monomorphisms.

Now suppose that

$$x_0, x_1 \in U((X_n)(m)) \coprod_{U((\tilde{L}_n(X_\bullet))(m))} U((\tilde{L}_n(Y_\bullet))(m))$$

satisfy  $\tilde{c}_n(m)(x_0) = \tilde{c}_n(m)(x_1)$ . Then, by the usual description of pushouts in the category of pointed sets as unions, there are three possibilities:

- **$x_0$  and  $x_1$  are both in  $U((X_n)(m))$ :** Since  $X_n \rightarrow Y_n$  is a flat cofibration, it is also a levelwise cofibration (see [9]), i.e., the map  $X_n(m) \rightarrow Y_n(m)$  is a cofibration in  $\mathcal{C}$  for all  $n$ . Hence  $U(X_n(m)) \rightarrow U(Y_n(m))$  is a monomorphism and hence  $x_0 = x_1$ .
- **$x_0$  and  $x_1$  are both in  $U((\tilde{L}_n(Y_\bullet))(m))$ :** Since  $Y_\bullet$  is Reedy levelwise-cofibrant, the map  $\tilde{L}_n(Y_\bullet) \rightarrow Y_n$  is a levelwise cofibration, i.e.,  $(\tilde{L}_n(Y_\bullet))(m) \rightarrow Y_n(m)$  is a cofibration in  $\mathcal{C}$  for all  $m$ , and consequently  $U(\tilde{L}_n(Y_\bullet)(m)) \rightarrow U(Y_n(m))$  is a monomorphism. Hence  $x_0 = x_1$ .
- **$x_0$  is in  $U(X_n(m))$  and  $x_1$  is in  $U((\tilde{L}_n(Y_\bullet))(m))$ :** (The same argument works in the case  $x_1$  is in  $U(X_n(m))$  and  $x_0$  is in  $U((\tilde{L}_n(Y_\bullet))(m))$ .) This part requires a bit more thought than the previous parts. Given any finitely complete, finitely cocomplete category  $\mathcal{A}$  and any simplicial object  $Z_\bullet$  of  $\mathcal{A}$ , the latching object  $\tilde{L}_n(Z_\bullet)$  of  $Z_\bullet$  is isomorphic, by Definition 2.1, to the coequalizer of a pair of maps whose codomain is a coproduct of  $n$  copies of  $Z_{n-1}$ , namely, one for each degeneracy map  $Z_{n-1} \rightarrow Z_n$ , and the domains as well as the maps themselves are built from finite limits and finite colimits of copies of  $Z_m$  for various  $m < n - 1$  and the degeneracy maps connecting them. If  $\mathcal{A}$  is the category of sets (or pointed sets), then  $\tilde{L}_n(Z_\bullet)$  is simply a coproduct of  $n$  copies of  $Z_{n-1}$  modulo equivalence relations coming from identifying subsets of the copies of  $Z_{n-1}$  given by intersections of copies of  $Z_m$  for  $m < n - 1$ . For each  $k \in \{0, \dots, n - 1\}$  we have a map  $\bar{d}_k : Z_n \rightarrow \tilde{L}_n Z_\bullet$  given by applying the face map  $d_k : Z_n \rightarrow Z_{n-1}$  and then including  $Z_{n-1}$  as the  $k$ th coproduct summand in  $\tilde{L}_n Z_\bullet = \left( \coprod_{i=0}^{n-1} Z_{n-1} \right) / \sim$ .

Now since taking the  $m$ -th “space” (really an object of  $\mathcal{C}$ ) is a functor from  $\mathbf{Sp}_{\mathcal{C}}$  to  $\mathcal{C}$ , applying the functor  $Z \mapsto Z(m)$  to a simplicial symmetric spectrum yields a simplicial object of  $\mathcal{C}$ . As limits and colimits in symmetric spectra are computed levelwise (see Example I.3.5 in [9]), this functor  $Z \mapsto Z(m)$  also commutes with limits and colimits. Let  $X_\bullet(m), Y_\bullet(m)$  denote the simplicial object of  $\mathcal{C}$  obtained by applying the  $m$ th “space” functor to  $X_\bullet$  and  $Y_\bullet$ , respectively. The fact that the  $m$ th “space” functor preserves limits and colimits now implies that the map  $(\tilde{L}_n(Y_\bullet))(m) \rightarrow Y_n(m)$  agrees, up to an isomorphism in the domain,

with the map  $\nu : \tilde{L}_n(Y_\bullet(m)) \rightarrow Y_n(m)$ . Since  $U$  preserves finite limits and finite colimits, the map  $U(\nu)$  agrees, up to an isomorphism in the domain, with the map  $\nu_Y : \tilde{L}_n(U(Y_\bullet)(m)) \rightarrow U(Y_n(m))$ . Similarly, applying  $U$  to the map  $\tilde{L}_n(X_\bullet(m)) \rightarrow X_n(m)$  yields, up to an isomorphism in the domain, the map  $\nu_X : \tilde{L}_n(U(X_\bullet)(m)) \rightarrow U(X_n(m))$ .

Here is the relevant consequence: we can choose an integer  $k \in \{0, \dots, n-1\}$  such that

$$x_1 \in U((\tilde{L}_n Y_\bullet)(m)) \cong \tilde{L}_n U((Y_\bullet)(m))$$

is in the  $k$ th coproduct summand in  $\tilde{L}_n U(Y_\bullet(m)) = \left(\coprod_{i=0}^{n-1} U(Y_{n-1}(m))\right) / \sim$ . Then  $\bar{d}_k(\nu_Y(x_1)) = x_1$  by design. Now the element  $\bar{d}_k(x_0) \in \tilde{L}_n(U(X_\bullet)(m))$  has two important features:

- we have equalities

$$\begin{aligned} (\tilde{L}_n(U(f_\bullet)(m)))(\bar{d}_k(x_0)) &= (\bar{d}_k(U(f_n)(m)))(x_0) \\ &= \bar{d}_k(\nu_Y(x_1)) \\ &= x_1, \end{aligned}$$

- and we have equalities

$$\begin{aligned} (U((f_n)(m)))(\nu_X(\bar{d}_k(x_0))) &= (\nu_Y(\tilde{L}_n(U(f_\bullet)(m)))(\bar{d}_k(x_0)) \\ &= \nu_Y(x_1) \\ &= U((f_n)(m))(x_0). \end{aligned}$$

Since each morphism of symmetric spectra  $f_n : X_n \rightarrow Y_n$  is a flat cofibration, it is also a levelwise cofibration (see Corollary 3.12 in Schwede's book [9]), hence each  $f_n(m)$  is a cofibration in  $\mathcal{C}$  and hence each  $U(f_n)(m)$  is a monomorphism, hence left-cancellable, and so

$$\nu_X(\bar{d}_k(x_0)) = x_0.$$

Now  $\nu_X(\bar{d}_k(x_0)) = x_0$  and  $(\tilde{L}_n(U(f_\bullet)(m)))(\bar{d}_k(x_0)) = x_1$  together imply that the element  $\bar{d}_k(x_0) \in \tilde{L}_n(U(X_\bullet)(m))$  maps to  $x_0$  and to  $x_1$  under the maps in the diagram

$$\begin{array}{ccc} \tilde{L}_n(U(X_\bullet)(m)) & \xrightarrow{\tilde{L}_n(U(f_\bullet)(m))} & \tilde{L}_n(U(Y_\bullet)(m)) \\ \nu_X \downarrow & & \nu_Y \downarrow \\ U((X_n)(m)) & \xrightarrow{U((f_n)(m))} & U(Y_n(m)) \end{array}$$

Consequently  $x_0$  and  $x_1$  represent the same element in the pushout

$$U\tilde{L}_n Y_\bullet(m) \coprod_{U\tilde{L}_n X_\bullet(m)} U(X_n(m)).$$

Consequently the map

$$\tilde{L}_n(U(Y_\bullet)(m)) \coprod_{\tilde{L}_n(U(X_\bullet)(m))} U(X_n(m)) \rightarrow U(Y_n(m))$$

given by the universal property of the pushout is injective. Since  $U$  commutes with finite limits and colimits, hence also with pushouts and with the formation of latching objects,



we get that the map given by the universal property of the pushout

$$U \left( \tilde{L}_n(Y_\bullet)(m) \coprod_{\tilde{L}_n(X_\bullet)(m)} X_n(m) \right) \rightarrow U(Y_n(m))$$

is also a monomorphism, hence that

$$\tilde{L}_n(Y_\bullet)(m) \coprod_{\tilde{L}_n(X_\bullet)(m)} X_n(m) \rightarrow Y_n(m)$$

is a cofibration in  $\mathcal{C}$  for each  $n$  and  $m$ . Hence

$$\tilde{L}_n(Y_\bullet) \coprod_{\tilde{L}_n(X_\bullet)} X_n \rightarrow Y_n$$

is a levelwise cofibration in  $\text{Sp}_{\mathcal{C}}$ , hence  $f_\bullet$  is a Reedy levelwise cofibration, as claimed.

If we furthermore assume that  $f_\bullet$  is a pointwise positive flat cofibration, then  $f_n(0) : X_n(0) \rightarrow Y_n(0)$  and  $L_n f(0) : L_n X(0) \rightarrow L_n Y(0)$  are isomorphisms for all  $n$ . Consequently

$$\tilde{L}_n(Y_\bullet)(0) \coprod_{\tilde{L}_n(X_\bullet)(0)} X_n(0) \rightarrow Y_n(0)$$

is an isomorphism, and consequently the canonical comparison map

$$\left( \tilde{L}_n(Y_\bullet) \coprod_{\tilde{L}_n(X_\bullet)} X_n \right)(0) \rightarrow Y_n(0)$$

is an isomorphism, which makes  $f_\bullet$  not only a Reedy levelwise cofibration but a Reedy *positive* levelwise cofibration.  $\square$

**Theorem 4.2.** *Let  $\mathcal{C}$  be a graded-concrete pointed model category satisfying the assumptions of Theorem 2.4, and let  $X_\bullet \xrightarrow{f_\bullet} Y_\bullet$  be a morphism of simplicial symmetric spectra in  $\mathcal{C}$ . Make the following assumptions:*

- *The map  $f_\bullet$  is a pointwise flat cofibration. That is, for each nonnegative integer  $n$ , the map  $X_n \xrightarrow{f_n} Y_n$  is a flat cofibration.*
- *Both  $X_\bullet$  and  $Y_\bullet$  are Reedy flat-cofibrant.*

*Then  $f_\bullet$  is a Reedy flat cofibration, and consequently the map of geometric realizations  $|f_\bullet| : |X_\bullet| \rightarrow |Y_\bullet|$  is a flat cofibration.*

*If we furthermore assume that  $f_\bullet$  is a pointwise positive flat cofibration, then  $f_\bullet$  is a Reedy positive flat cofibration, hence  $|f_\bullet|$  is a positive flat cofibration.*

*Proof.* Let  $\tilde{P}\tilde{O}^n$  denote the symmetric spectrum in  $\mathcal{C}$  defined to be the pushout in the square

$$\begin{array}{ccc} \tilde{L}_n(X_\bullet) & \xrightarrow{\tilde{L}_n(f_\bullet)} & \tilde{L}_n(Y_\bullet) \\ \downarrow \tilde{v}_n & & \downarrow \\ X_n & \longrightarrow & \tilde{P}\tilde{O}^n, \end{array}$$

and let  $PO(n, m)$  denote the object of  $\mathcal{C}$  defined to be the pushout in the square

$$\begin{array}{ccc} L_m(\tilde{P}\tilde{O}^n) & \longrightarrow & L_m(Y_n) \\ \downarrow v_n & & \downarrow \\ \tilde{P}\tilde{O}^n(m) & \longrightarrow & PO(n, m). \end{array}$$

We need to show that the canonical map  $c_{n,m} : PO(n, m) \rightarrow Y_n(m)$ , given by the universal property of the pushout, is a cofibration in  $\mathcal{C}$  for all nonnegative integers  $m$  and  $n$ . Indeed, fix  $n$ , and suppose we have shown that  $c_{n,m}$  is a cofibration for all values of  $m$ . This is exactly the condition required for the canonical map  $\tilde{P}\mathcal{O}^n \rightarrow Y_n$ , given by the universal property of the pushout, to be a flat cofibration. If we show that this canonical map is a flat cofibration for all  $n$ , then we have shown  $f_\bullet$  a Reedy flat cofibration, by definition.

To show that  $c_{n,m}$  is a cofibration, we explicitly check that, after applying a functor  $U$  as in Definition 3.1, the map  $Uc_{n,m}$  is injective on simplices. *Throughout, we freely make use of the fact that  $U$  preserves finite limits and finite colimits, and consequently sends latching objects to latching objects*, as discussed in the proof of Theorem 4.1. Suppose that  $x, y$  are simplices in  $U(PO(n, m))$  such that  $Uc_{n,m}(x) = Uc_{n,m}(y)$ . Then, since pushouts in simplicial sets are computed pointwise, there are three possibilities:

- **$x, y$  are the images of simplices  $\bar{x}, \bar{y} \in U(L_m(Y_n))$  under the map  $U(L_m(Y_n)) \rightarrow U(PO(n, m))$ :**  $Y_\bullet$  is assumed to be Reedy cofibrant, hence is levelwise flat-cofibrant, hence  $Y_n$  is flat-cofibrant and hence the map  $U(L_m(Y_n)) \rightarrow U(Y_n(m))$  is a monomorphism of graded pointed sets for all  $m$ , and consequently  $x = y$ .
- **$y$  is the image of some simplex  $\bar{y}$  under the map  $U(L_m(Y_n)) \rightarrow U(PO(n, m))$  and  $x$  is the image of some simplex  $\bar{x}$  under the map  $U(\tilde{P}\mathcal{O}^n(m)) \rightarrow U(PO(n, m))$ :** Then there are two sub-cases to consider:

- **$x$  is the image of some simplex  $\bar{x}' \in U(X_n(m))$  under the map  $U(X_n(m)) \rightarrow U(\tilde{P}\mathcal{O}^n(m))$ :** Since  $f_\bullet$  is assumed a levelwise flat cofibration, the map  $X_n \xrightarrow{f_n} Y_n$  is a flat cofibration, and hence the map

$$U(X_n(m)) \coprod_{U(L_m(X_n))} U(L_m(Y_n)) \rightarrow U(Y_n(m))$$

is a monomorphism of graded pointed sets. This map factors through a map

$$U(X_n(m)) \coprod_{U(L_m(X_n))} U(L_m(Y_n)) \rightarrow U(PO(n, m))$$

which is also a monomorphism since it is the first map in a composite map that is a monomorphism. By commutativity of the relevant diagrams, this implies that  $x = y$ .

- **$x$  is the image of some simplex  $\bar{x}' \in U((\tilde{L}_n(Y_\bullet))(m))$  under the map  $U((\tilde{L}_n(Y_\bullet))(m)) \rightarrow U(\tilde{P}\mathcal{O}^n(m))$ :** Then  $\bar{x}', \bar{y}'$  each define an element  $x', y'$  in the pushout set

$$U\left(\left(\tilde{L}_n(Y_\bullet)(m) \coprod_{L_m(\tilde{L}_n(Y_\bullet))} L_m(Y_n)\right)\right),$$

and these two elements map to the same element of  $U(Y_n(m))$ , since  $c_{n,m}(x) = c_{n,m}(y)$ . Since  $Y_\bullet$  is Reedy flat-cofibrant, the map of graded pointed sets

$$U\left(\left(\tilde{L}_n(Y_\bullet)(m) \coprod_{L_m(\tilde{L}_n(Y_\bullet))} L_m(Y_n)\right)\right) \rightarrow U(Y_n(m))$$

is a monomorphism; consequently  $x' = y'$ , and hence  $x', y'$  both pull back to a single element  $u \in U(L_m(\tilde{L}_n(Y_\bullet)))$ , and the image of this element under the map

$$U(L_m(\tilde{L}_n(Y_\bullet))) \rightarrow U(PO(n, m))$$

is equal to both  $x$  and  $y$ ; hence  $x = y$ .

- $x, y$  are the images of simplices  $\bar{x}, \bar{y} \in U(\tilde{P}O^n(m))$  under the map

$$U(\tilde{P}O^n(m)) \rightarrow U(PO(n, m))$$

By Theorem 4.1 the map  $\tilde{P}O^n \rightarrow Y_n$  is a levelwise cofibration so  $U((\tilde{P}O^n)_m) \rightarrow U(Y_n(m))$  is a monomorphism and therefore  $\bar{x} = \bar{y}$ , which implies  $x = y$ .

The above argument shows that the canonical comparison map

$$c_{n,m} : PO(n, m) = L_m(Y_n) \coprod_{L_m(\tilde{P}O^n)} \tilde{P}O^n(m) \rightarrow Y_n(m)$$

is a cofibration in  $\mathcal{C}$  for all  $m$  and  $n$ , hence that the canonical map of symmetric spectra

$$(4.0.3) \quad \tilde{L}_n(Y_\bullet) \coprod_{\tilde{L}_n(X_\bullet)} X_n = \tilde{P}O^n \rightarrow Y_n$$

is a flat cofibration for all  $n$ , hence that  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  is a Reedy flat cofibration. If we furthermore assume that  $f_\bullet$  is a pointwise *positive* flat cofibration, then by Theorem 4.1, the map 4.0.3 is a positive levelwise cofibration in addition to being a flat cofibration; so  $f_\bullet$  is a Reedy positive flat cofibration, as claimed.  $\square$

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