ALGEBRAIC K-THEORY AND CHROMATIC HOMOTOPY THEORY

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These are notes based on a talk I gave at the AIM Workshop on "Equivariant techniques in stable homotopy theory" in May 2021.

1. Algebraic K-theory

Classically, algebraic K-theory assigns a graded abelian group $\mathsf{K}_*(R)$ to a ring R. The insight of Quillen, Segal, and Waldhausen is that the correct input is actually a category $\mathcal{C}$ with some notion of exact sequences and equivalences and the correct output is a spectrum $\mathsf{K}(\mathcal{C})$. In modern formulations, we view it as a functor

$$\mathsf{K}\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathsf{Sp}$$

from the ∞-category of small stable ∞-categories to the ∞-category of spectra. The main example of interest today is $\mathsf{K}(R) := \mathsf{K}(\mathrm{Perf}(R))$ where R is an E_1 ring.

Rather than give an explicit construction of algebraic K-theory, I will say why we care about it. In short, my answer to this question is that it encodes deep information about

(1) the arithmetic of E_1 rings, and
(2) the geometry of manifolds.

I will focus more on the first. Before we begin, I fix notation from chromatic homotopy theory:

(1) Fix a prime p,
(2) Let $K(n)$ denote Morava K-theory with coefficients $\mathbb{F}_p[v_n^{\pm 1}]$. These are the *prime fields* in spectra.
(3) Let $F(n-1)$ denote a finite type n spectrum, such as $V(n-1) = \mathbb{S}/(p, v_1, \dots v_{n-1})$ when it exists, ie $F(n-1)$ is a finite cell $\mathbb{S}$ module such that

$$K(n-1)_*F(n-1) = 0 \text{ and } K(n)_*F(n-1) \neq 0.$$

Such spectra exist for each n by a result of J.H. Smith [HS98, Theorem 4.10].
(4) Let $T(n) = v_n^{-1}F(n-1)$, which also exist by [HS98, Theorem 9].
(5) Write $L_n = L_{K(0)\oplus\cdots\oplus K(n)}$ and $L_n^f = L_{T(0)\oplus \dots T(n)}$.

2. Algebraic K-theory and arithmetic of E_1 rings

2.1. **Descent.** One reason why we like algebraic algebraic K-theory is that it satisfies *Nisnevich descent*. In particular, given a smooth scheme X there is a descent spectral sequence

$$\mathrm{H}^{-s}_{\mathrm{mot}}(X; \mathbb{Z}(t/2)) \implies \pi_{s+t}\,\mathsf{K}(X)$$

which we may view as an Atiyah–Hirzebruch spectral sequence for algebraic K-theory as functor K as a homology theory on the category of smooth schemes. Of particular interest is the case $X = \mathrm{spec}(\mathcal{O}_F)$ where F is the ring of integers in a number field F with $1/p \in \mathcal{O}_F$.

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2.2. **LQ conjecture.** Stephan Lichtenbaum [Lic73] and Daniel Quillen [Qui75] each made several related conjectures about the relationship between algebraic K-theory and étale cohomology. One version of one of these conjectures is the following.

Conjecture 2.1 (LQC)**.** *For a finitely generated $\mathbb{Z}$-algebra A with $1/p \in A$, is that there is a descent spectral sequence*

$$\mathrm{H}^{-s}_{\text{ét}}(\operatorname{spec}(A); \mathbb{Z}_p(t/2)) \implies \pi_{s+t} \mathsf{K}(A)_p$$

which converges for $s + t > dim(A) + 1$?

For simplicity, let $X = \operatorname{spec}(\mathcal{O}_F[1/p])$ where F is a number field and $\mathcal{O}_F$ is its ring of integers. Thomason [Tho85] proved that there is a strongly convergent spectral sequence. Combining rigidity results of Suslin [Sus83] and Gabber [Gab92] with the Norm-Residue theorem of Voevodsky and Rost [Voe11], we know that there is an isomorphism

$$\mathrm{H}^{-s}_{\text{mot}}(X; \mathbb{Z}_p(t/2)) \cong \mathrm{H}^{-s}_{\text{ét}}(X; \mathbb{Z}_p(t/2))$$

for $s + t >> 0$, implying the LQC. This case is particularly interesting, because of Lichtenbaum's conjecture that

$$|\mathsf{K}_{4k-1}(\mathcal{O}_F)|/|\mathsf{K}_{4k-2}(\mathcal{O}_F)| = \pm 2^{r(F)} \cdot \zeta_F(-2k-1)$$

and Wiles' [Wil90] proof of the Iwasawa main conjecture, which relates quotients of orders of étale cohomology groups to special values of Dedekind zeta functions.

2.3. **ARLQ red-shift conjecture.** In Waldhausen's formulation [Wal84], the LQC is the conjecture that the map

$$\pi_* \mathsf{K}(X)_p \to \pi_* L_1^f \mathsf{K}(X)_p$$

is co-connective. In the same spirit, Rognes has posed the following higher chromatic height Lichtenbaum-Quillen conjecture:

Conjecture 2.2 (ARLQ red-shift conjecture [Rog14b, Conj. 4.1],[Rog])**.** *If R is an E_1 ring and the map*

$$R_p \to L_n^f R_p$$

is co-connective (the fiber is co-connective) then is the map

$$\mathsf{K}(R)_p \to L_{n+1}^f \mathsf{K}(R)_p$$

co-connective?

Remark 2.3. This is not exactly how Rognes phrases it, but it is implied by the FP-type red-shift conjecture (a slightly weaker form of the pure FP-type red-shift conjecture in [Rog]) by [MR99, Thm. 8.2] as explained in [HW20, Cor. 6.0.5]. The phrasing in [Rog14b, Conj. 4.1] is very close to this phrasing.

Example 2.4. The following examples are known:

(1) (Case n=-1) When $R = \mathbb{F}_p$, a consequence of Quillen [Qui72] is the equivalence $\mathsf{K}(\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$
(2) (Case n=0) When $R = \mathcal{O}_F[1/p]$, this is a consequence of the proof of the LQC.
(3) (Case n=1) When $R = \ell$, L, ku, KU, $k(1)$, and $K(1)$, then

$$\mathsf{K}(R)_p \to L_2^f \mathsf{K}(R)_p$$

is co-connective by explicit calculations of Ausoni and Ausoni–Rognes and localization sequence of Blumberg–Mandell [Aus10, AR02, AR12, BM08].
(4) (Case $n \geq 2$) When $R = BP\langle n \rangle$, the map

$$\mathsf{K}(R)_p \to L_{n+1}^f \mathsf{K}(R)_p$$

is co-connective by [HW20].

Problem 2.5. Can we prove the ARLQ red-shift conjecture for other families of spectra?

As a particular example of interest, the following was (essentially) conjectured by Ausoni–Rognes:

Conjecture 2.6 ([AR08, Conj. 4.3])**.** *The map*

$$\mathsf{K}(L_{K(n)}\mathbb{S})_p \to L^f_{n+1}\,\mathsf{K}(L_{K(n)}\mathbb{S})_p$$

is co-connective.

Remark 2.7. Moreover, in [Rog14a], Rognes proposed a spectral sequence

$$\mathrm{H}^{-s}_{\text{ét}}(\operatorname{spec}(BP\langle 1\rangle_p);\mathbb{F}_{p^2}(t/2)) \implies \pi_{s+t}V(1)\wedge \mathsf{K}(BP\langle 1\rangle_p)$$

converging in sufficiently large degrees, analoguous to the spectral sequence of the LQC. The calculation of Ausoni–Rognes shows that

$$V(1)_*\,\mathsf{K}(BP\langle 1\rangle_p) \cong \mathbb{F}_p[v_2]\otimes B_2$$

where B_2 is finite graded $\mathbb{F}_p$ vector space. Here $\mathbb{F}_{p^2}(t/2) = V(1)_*E_2$, so again by analogy with the classical setting, one expects that

$$V(1)_*E_2 \cong V(1)_*\,\mathsf{K}(\Omega_1)$$

where Ω_1 is a maximal extension of the fraction field ffL_p.

Problem 2.8. Compute

$$\mathrm{H}^{-s}_{\text{ét}}(\operatorname{spec}(BP\langle n\rangle_p);F(n)_*E_{n+1}).$$

Is it always a finitely generated free $P(v_n+1)$ module? Can we prove that

$$F(n)_*\,\mathsf{K}(\Omega_n)\cong F(n)_*E_{n+1}$$

more generally where Ω_n is a maximal extension of the fraction field of $BP\langle n\rangle_p$?

2.4. **Morava K-theory vanishing.** A slightly weaker form of the ARLQ red-shift conjecture is the following.

Conjecture 2.9 (Red-shift philosophy)**.** *Suppose R is an E_1 ring, then*

$$K(m)_*R = 0 \textit{ if and only if } K(m+1)_*\,\mathsf{K}(R) = 0$$

Definition 2.10. We say R has height n if

$$n = \max\{m : K(m)_*R \neq 0\}$$

Remark 2.11. When R is an E_∞ ring, then $K(n+1)_*R = 0$ implies $K(m)_*R = 0$ for $m \geq n+2$ by a result of Hahn [Hah16], so this notion of height is a good notion of chromatic complexity for E_∞ rings. For E_1 rings, it is not as well behaved of a notion.

If R is an E_∞ ring and it has height $\leq n$, we may ask if $K(R)$ has height $\leq n+1$, which is one direction of the red-shift philosophy.

Example 2.12. The following examples of such phenomena are known (in chronological order):

(1) (Case $n=-1$) Quillen [Qui72] showed $\mathsf{K}(\mathbb{F}_p)_p = H\mathbb{Z}_p$ and this implies $L_{K(m)}\,\mathsf{K}(\mathbb{F}_p) = 0$ for $m \geq 1$. The same result holds for any $\mathbb{F}_p$-algebra.

(2) (Case $n=0$) Mitchell proved that

$$L_{K(m)}K(\mathbb{Z}) = 0$$

for $m \geq 2$. The same result holds for any $H\mathbb{Z}$ algebra or scheme X.

(3) (Case $n=1$) Ausoni–Rognes proved that

$$L_{K(m)}K(\ell) = 0$$

for $m \geq 3$. The same result holds for any ℓ algebra.

(4) (Case n=2) A-K–Salch proved that

$$L_{K(m)}\,\mathsf{K}(BP\langle 2\rangle) = 0$$

for $m \geq 4$ at $p = 2,3$. The same result holds for any $BP\langle n\rangle$ algebra.

(5) (Arbitrary n) A-K–Quigley prove that $K(m)_*y(n) = 0$ implies that $K(m)_*\,\mathrm{TP}(y(n) = 0$.

(6) (Arbitrary n) Clausen–Mathew–Naumann–Noel and Land–Mathew–Meier-Tamme show that when R is an E_1 ring and $L_{T(n-1)\oplus T(n)}R = 0$, then $L_{T(n)}\,\mathsf{K}(R) = 0$.

2.5. **Morava K-theory non-vanishing and LQC at arbitrary heights.** Another reason we like algebraic K-theory is that is has a universal property. In particular, it is the universal additive invariant: for any functor

$$E\colon\ \mathsf{Cat}_{\infty}^{\mathrm{ex}} \to \mathsf{Sp}$$

that commutes with filtered colimits, inverts Morita equivalences, and sends split exact sequences of stable ∞-categories to split fiber sequences. In particular, we get trace maps

$$\mathsf{K} \Rightarrow \mathrm{TC} \Rightarrow \mathrm{TC}^{-} \to \mathrm{THH}\,.$$

When R is an E_∞-ring spectrum, we can also let $\mathsf{K}^{(n)}(R) = \mathsf{K}(\dots K(R)\dots)$ and similarly consider

$$\mathsf{K}^{(n)}(R) \to (\mathbb{T}^n \otimes R)^{h\mathbb{T}^n}.$$

The idea is that even though $\mathsf{K}^{(n)}(R)$ is rarely complex oriented, we can sometimes show that $(\mathbb{T}^n \otimes R/\,\mathrm{MU})^{h\mathbb{T}^n}$ is an even complex oriented ring spectrum or in the case of an E_∞ ring map $A \to B$ where B is an even complex oriented ring spectrum, we can map to $B^{h\mathbb{T}^n}$ and detect higher chromatic height information there.

Example 2.13. Again we list known examples

(1) (case n=-1) [Gro57] $\mathbb{Q} \otimes \mathsf{K}(\mathbb{F}_p) \neq 0$
(2) (case n=0) [Voe11] $K(1)_*\,\mathsf{K}(\mathcal{O}_F[1/p]) \neq 0$
(3) (case n=1) [Aus10, AR02, BM08] $R = \ell, ku, k(1), \ell, KU, K(1)$, then $K(m)_*\,\mathsf{K}(R) \neq 0$ for $0 \leq m \leq 2$.
(4) (arbitrary height) [HW20] When $0 \leq m \leq n+1$, then
$$L_{K(m)}\,\mathsf{K}(BP\langle n\rangle) \neq 0$$
for $0 \leq m \geq n+1$.
(5) (arbitrary height) [Yua21] When F is a field with $1/p \in F$
$$L_{K(m)}\,\mathsf{K}^{(n)}(F) \neq 0$$
for $0 \leq m \leq n$. When E^{tC_p} is $K(n)$-local, then
$$L_{K(n+1)}\,\mathsf{K}(E^{tC_p}) \neq 0.$$
(6) (height 2) [**?**] A related result of a different flavor, is the result that
$$\mathsf{K}(\mathsf{K}(\mathbb{Z}))$$
detects a family of β-elements in $\pi_*\mathbb{S}$ at $p \geq 5$. This was proven by detecting these elements in $\mathrm{TC}^{-}_*(j)$ where $j = \tau_{\geq 0} L_{K(1)}\mathbb{S}$.

Problem 2.14. Can we draw explicit connections between $\mathsf{K}^{(2)}(\mathcal{O}_F)$ and modular forms over $\mathcal{O}_F$? What if we replace iterated algebraic K-theory with secondary K-theory?

Problem 2.15. Can we prove similar results for other families of E_n rings?

2.6. **Galois descent.** We say a map $A \to B$ of E_∞ rings is an E G-Galois extension if G acts on B by A-algebra maps and

(1) the map $A \to B^{hG}$ is a E-local equivalence, and
(2) the map $B \wedge_A B \to F(G_+, B)$ is a E-local equivalence.

Inspired by the Galois descent conjecture of Lichtenbaum, Ausoni–Rognes conjectured.

Conjecture 2.16 (Ausoni–Rognes Galois descent conjecture)**.** *Suppose $A \to B$ is a $K(n)$-local G-Galois extension, then*

$$T(n+1) \wedge \mathsf{K}(A) \to T(n+1) \wedge \mathsf{K}(A)^{hG}$$

is $K(n)$-local equivalence.

Example 2.17. We again list some examples:

(1) (Case $n = 0$) Galois descent conjecture of Lichtenbaum.

(2) (Case $n = 1$) For $L_p \to KU_p$, by [Aus10, AR02, BM08].
(3) (Arbitrary heights) When $A \to B$ is a G Galois extension for a finite group G such that the image of the transfer $\mathsf{K}_0(B)\mathbb{Q} \to \mathsf{K}_0(A) \otimes \mathbb{Q}$ contains the unit, then

$$L_n^f \mathsf{K}(A) \to L_n^f \mathsf{K}(B)^{hG}$$

is an equivalence by [CMNN20]. For example the C_2 Galois extension $KO \to KU$, the $GL_2(\mathbb{Z}/n)$ Galois extension $\mathrm{TMF}[1/n] \to \mathrm{TMF}(n)$, $E(G,k)^{hH} \to E(G,k)$ is an H Galois extension where $E(G,k)$ is higher real K-theory associated to a formal group G over a perfect field k of characteristic p and H is a finite subgroup of the automorphisms $\mathrm{Aut}(G,k)$ of k.

Problem 2.18. Does algebraic K-theory satisfy *pro-finite* Galois descent? For example, is there a spectral sequence

$$\mathrm{H}^*(\mathbb{G}_n; T(n+1)_* \mathsf{K}(E_n)) \implies T(n+1)_* \mathsf{K}(L_{K(n)}\mathbb{S})?$$

Can condensed/pycknotic mathematics of Clausen–Scholze/Barwick–Haine be used here?

3. Stable diffeomorphisms of manifolds

We also care about algebraic K-theory because of applications to geometry. In particular, for a smooth manifold M,

$$\mathsf{K}(\mathbb{S}[\Omega M]) = \mathbb{S}[M] \oplus \mathrm{Wh}^{\mathrm{DIFF}}(M)$$

where $\mathrm{Wh}^{\mathrm{DIFF}}(M)$ encodes stable diffeomorphisms of M. This is already interesting when M is a point. Waldhausen [Wal84] suggested a program for computing algebraic K-theory of of the sphere spectrum by working up the tower

$$\mathsf{K}(\mathbb{S}_{(p)}) \longrightarrow \cdots \longrightarrow \mathsf{K}(L_2^f\mathbb{S}_{(p)}) \longrightarrow \mathsf{K}(L_1^f\mathbb{S}) \longrightarrow \mathsf{K}(H\mathbb{Q}).$$

By [MS93], the map

$$\mathsf{K}(\mathbb{S}_{(p)}) \to \operatorname*{holim}_n \mathsf{K}(\tau_{\geq 0} L_n^f \mathbb{S})$$

is an equivalence, an analogue of the chromatic convergence theorem. We also know $\mathsf{K}_*(\mathbb{S})$ in a range by work of Rognes [Rog03] and Mandell–Blumberg, but we still know less about $\mathsf{K}_*(\mathbb{S})$ then $\pi_*\mathbb{S}$ (and in fact $\pi_*\mathbb{S}$ is a summand).

Another approach to algebraic K-theory of the sphere spectrum was suggested by Dundas–Rognes [DR18]. Consider the Amitsur complex

$$\mathbb{S} \longrightarrow \mathrm{MU} \rightrightarrows \mathrm{MU}^{\otimes 2} \Rrightarrow \cdots,$$

and apply algebraic K-theory. Then [DR18], prove that algebraic K-theory satisfies co-simplicial descent so that

$$\mathsf{K}(\mathbb{S}) \simeq \mathrm{Tot}(\mathsf{K}(\mathrm{MU}^{\otimes \bullet})).$$

Alternatively, when $A \to B$ is a G Galois extension for a finite group, so that $B \wedge_A B \simeq F(G_+, B)$, we produces a spectral sequence

$$H^*(G; \mathsf{K}_*(B))) \implies \mathsf{K}_*(B)^{hG}$$

and one can combine the results of Dundas–Rognes [DR18] with the results of Clausen–Mathew–Naumann–Noel [CMNN20] to compute $\mathsf{K}_*(A)$ in some cases.

Problem 3.1. Compute $\mathsf{K}_*(\mathrm{MU})$ and $\mathsf{K}_*(L_n^f\mathbb{S})$. Use Galois descent for a G Galois extension $A \to B$ to compute algebraic K-theory of A from algebraic K-theory of B, for example compute $F(1)_* \mathsf{K}(KO)$ by computing $F(1)_* \mathsf{K}(KU)$ at $p = 2$ and using descent.

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