ALGEBRAIC K-THEORY AND CHROMATIC HOMOTOPY THEORY

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These are notes based on a talk I gave at the AIM Workshop on "Equivariant techniques in stable homotopy theory" in May 2021. (These notes were revised minimally in March 2022. If you notice typos or inconsistencies, please feel free to email me.)

1. Algebraic K-theory

Classically, algebraic K-theory assigns a graded abelian group $\mathsf{K}_*(R)$ to a ring R. The insight of Quillen, Segal, and Waldhausen is that the correct input is actually a category $\mathcal{C}$ with some notion of exact sequences and equivalences and the correct output is a spectrum $\mathsf{K}(\mathcal{C})$. In modern formulations, we view it as a functor

$$\mathsf{K}\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathsf{Sp}$$

from the ∞-category of small stable ∞-categories to the ∞-category of spectra. The main example of interest today is $\mathsf{K}(R) := \mathsf{K}(\mathrm{Perf}(R))$ where R is an $\mathbb{E}_1$-ring.

Rather than give an explicit construction of algebraic K-theory, I will say why we care about it. In short, my answer to this question is that it encodes deep information about

(1) the arithmetic of $\mathbb{E}_1$-rings, and
(2) the geometry of manifolds.

I will focus more on the first. Before we begin, I fix notation from chromatic homotopy theory:

(1) Fix a prime p,
(2) Let $K(n)$ denote Morava K-theory with coefficients $\mathbb{F}_p[v_n^{\pm 1}]$. These are the *prime fields* in spectra.
(3) Let $F(n-1)$ denote a finite, type n spectrum, such as $V(n-1) = \mathbb{S}/(p, v_1, \dots v_{n-1})$ when it exists, i.e. $F(n-1)$ is a finite cell $\mathbb{S}$ module such that

$$K(n-1)_*F(n-1) = 0 \text{ and } K(n)_*F(n-1) \neq 0.$$

Such spectra exist for each n by a result of J.H. Smith [HS98, Theorem 4.10].
(4) Let $T(n) = v_n^{-1}F(n-1)$, which also exist by [HS98, Theorem 9].
(5) Write $L_n = L_{K(0)\oplus\cdots\oplus K(n)}$ and $L_n^f = L_{T(0)\oplus\cdots\oplus T(n)}$.

2. Algebraic K-theory and arithmetic of $\mathbb{E}_1$-rings

2.1. **Descent.** One reason why we like algebraic algebraic K-theory is that it satisfies *Nisnevich descent*. In particular, given a smooth scheme X there is a descent spectral sequence

$$\mathrm{H}^{-s}_{\mathrm{mot}}(X; \mathbb{Z}(t/2)) \implies \pi_{s+t}\,\mathsf{K}(X)$$

which we may view as an Atiyah–Hirzebruch spectral sequence for algebraic K-theory, where algebraic K-theory is viewed as a homology theory on the category of smooth schemes. Of particular interest is the case $X = \mathrm{spec}(\mathcal{O}_F)$ where F is the ring of integers in a number field F with $1/p \in \mathcal{O}_F$.

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2.2. **LQ conjecture.** Stephan Lichtenbaum [Lic73] and Daniel Quillen [Qui75] each made several related conjectures about the relationship between algebraic K-theory and étale cohomology. One version of one of these conjectures is the following.

Conjecture 2.1 (LQC)**.** *For a finitely generated $\mathbb{Z}$-algebra A with $1/p \in A$, is there a descent spectral sequence*

$$\mathrm{H}^{-s}_{\text{ét}}(\operatorname{spec}(A); \mathbb{Z}_p^{\wedge}(t/2)) \implies \pi_{s+t}\,\mathsf{K}(A)_p^{\wedge}$$

which converges for $s+t > \dim(A)+1$?

For simplicity, let $X = \operatorname{spec}(\mathcal{O}_F[1/p])$ where F is a number field and $\mathcal{O}_F$ is its ring of integers. Thomason [Tho85] proved that there is a strongly convergent spectral sequence.

$$\mathrm{H}^{-s}_{\text{ét}}(\operatorname{spec}(A); \mathbb{Z}_p^{\wedge}(t/2)) \implies \pi_{s+t}L_1^f\,\mathsf{K}(A)_p^{\wedge}$$

Combining rigidity results of Suslin [Sus83] and Gabber [Gab92] with the Norm-Residue theorem of Voevodsky and Rost [Voe11], we know that there is an isomorphism

$$\mathrm{H}^{-s}_{\text{mot}}(X; \mathbb{Z}_p(t/2)) \cong \mathrm{H}^{-s}_{\text{ét}}(X; \mathbb{Z}_p(t/2))$$

for $s+t >> 0$, implying the LQC. This case is particularly interesting, because of Lichtenbaum's conjecture that

$$|\,\mathsf{K}_{4k-1}(\mathcal{O}_F)|/|\,\mathsf{K}_{4k-2}(\mathcal{O}_F)| = \pm 2^{r(F)} \cdot \zeta_F(-2k-1)$$

and Wiles' [Wil90] proof of the Iwasawa main conjecture, which relates quotients of orders of étale cohomology groups to special values of Dedekind zeta functions.

2.3. **ARLQ red-shift conjecture.** In Waldhausen's formulation [Wal84], the LQC is the conjecture that the map

$$\pi_*\,\mathsf{K}(X)_p \to \pi_* L_1^f\,\mathsf{K}(X)_p$$

is co-connective. In the same spirit, Rognes has posed the following higher chromatic height Lichtenbaum-Quillen conjecture:

Conjecture 2.2 (ARLQ red-shift conjecture [Rog14b, Conj. 4.1],[Rog])**.** *If R is an E_1 ring and the map*

$$R_p \to L_n^f R_p$$

is co-connective (the fiber is co-connective) then is the map

$$\mathsf{K}(R)_p \to L_{n+1}^f\,\mathsf{K}(R)_p$$

co-connective?

Remark 2.3. This is not exactly how Rognes phrases it, but it is implied by the FP-type red-shift conjecture (a slightly weaker form of the pure FP-type red-shift conjecture in [Rog]) by [MR99, Thm. 8.2] as explained in [HW20, Cor. 6.0.5]. The phrasing in [Rog14b, Conj. 4.1] is very close to this phrasing.

Example 2.4. The following examples are known:

(1) (Case n=-1) When $R = \mathbb{F}_p$, a consequence of Quillen [Qui72] is the equivalence $\mathsf{K}(\mathbb{F}_p)_p^{\wedge} \simeq H\mathbb{Z}_p^{\wedge}$
(2) (Case n=0) When $R = \mathcal{O}_F[1/p]$, this is a consequence of the proof of the LQC.
(3) (Case n=1) When $R = \ell$, L, ku, KU, $k(1)$, and $K(1)$, then

$$\mathsf{K}(R)_p \to L_2^f\,\mathsf{K}(R)_p$$

is co-connective by explicit calculations of Ausoni and Ausoni–Rognes and localization sequence of Blumberg–Mandell [Aus10, AR02, AR12, BM08].
(4) (Case $n \geq 2$) When $R = BP\langle n\rangle$, the map

$$\mathsf{K}(R)_p \to L_{n+1}^f\,\mathsf{K}(R)_p$$

is co-connective by [HW20].

Problem 2.5. Can we prove the ARLQ red-shift conjecture for other families of spectra?

As a particular example of interest, the following was (essentially) conjectured by Ausoni–Rognes:

Conjecture 2.6 ([AR08, Conj. 4.3])**.** *The map*

$$\mathsf{K}(L_{K(n)}\mathbb{S})_p \to L^f_{n+1}\,\mathsf{K}(L_{K(n)}\mathbb{S})_p$$

is co-connective.

Remark 2.7. Moreover, in [Rog14a], Rognes proposed a spectral sequence

$$\mathrm{H}^{-s}_{\text{ét}}(\mathrm{spec}(BP\langle 1\rangle^{\wedge}_p);\mathbb{F}_{p^2}(t/2)) \implies \pi_{s+t}V(1)\wedge \mathsf{K}(BP\langle 1\rangle^{\wedge}_p)$$

converging in sufficiently large degrees, analogous to the spectral sequence of the LQC. The calculation of Ausoni–Rognes [AR02] shows that

$$V(1)_*\,\mathrm{TC}(BP\langle 1\rangle) \cong \mathbb{F}_p[v_2]\otimes B_2$$

where B_2 is finite graded $\mathbb{F}_p$ vector space. Here $\mathbb{F}_{p^2}(t/2) = V(1)_*E_2$, so again by analogy with the classical setting, one expects that

$$V(1)_*E_2 \cong V(1)_*\,\mathsf{K}(\Omega_1)$$

where Ω_1 is a maximal extension of the fraction field ffL_p.

Problem 2.8. Compute

$$\mathrm{H}^{-s}_{\text{ét}}(\mathrm{spec}(BP\langle n\rangle^{\wedge}_p);F(n)_*E_{n+1}).$$

Is $F(n)_*\,\mathrm{TC}(BP\langle n\rangle)$ always a finitely generated free $P(v_{n+1})$ module? Can we prove that

$$F(n)_*\,\mathsf{K}(\Omega_n) \cong F(n)_*E_{n+1}$$

more generally where Ω_n is a maximal extension of the fraction field of $BP\langle n\rangle^{\wedge}_p$?

2.4. **Morava K-theory vanishing.** A slightly weaker form of the ARLQ red-shift conjecture is the following.

Conjecture 2.9 (Red-shift philosophy)**.** *Suppose R is an $\mathbb{E}_1$-ring, then*

$$K(m)_*R = 0 \textit{ if and only if } K(m+1)_*\,\mathsf{K}(R) = 0$$

Definition 2.10. We say R has height n if

$$n = \max\{m : K(m)_*R \neq 0\}$$

Remark 2.11. When R is an $\mathbb{E}_\infty$-ring, then $K(n+1)_*R = 0$ implies $K(m)_*R = 0$ for $m \geq n+2$ by a result of Hahn [Hah16], so this notion of height is a good notion of chromatic complexity for $\mathbb{E}_\infty$-rings. For $\mathbb{E}_1$-rings, it is not as well behaved of a notion.

If R is an $\mathbb{E}_\infty$-ring and it has height $\leq n$, we may ask if $\mathsf{K}(R)$ has height $\leq n+1$, which is one direction of the red-shift philosophy.

Example 2.12. The following examples of such phenomena are known (in chronological order):

(1) (Case $n = -1$) Quillen [Qui72] showed $\mathsf{K}(\mathbb{F}_p)_p = H\mathbb{Z}_p$ and this implies $L_{K(m)}\,\mathsf{K}(\mathbb{F}_p) = 0$ for $m \geq 1$. The same result holds for any $\mathbb{F}_p$-algebra.

(2) (Case $n = 0$) Mitchell proved that

$$L_{K(m)}K(\mathbb{Z}) = 0$$

for $m \geq 2$. The same result holds for any $H\mathbb{Z}$ algebra or scheme X.

(3) (Case $n = 1$) Ausoni–Rognes proved that

$$L_{K(m)}K(\ell) = 0$$

for $m \geq 3$. The same result holds for any ℓ algebra.

(4) (Case n=2) A-K–Salch proved that
$$L_{K(m)}\,\mathsf{K}(BP\langle 2\rangle)=0$$
for $m\geq 4$ at $p=2,3$. The same result holds for any $BP\langle n\rangle$ algebra.
(5) (Arbitrary n) A-K–Quigley prove that $K(m)_*y(n)=0$ implies that $K(m+1)_*\operatorname{TP}(y(n)=0$.
(6) (Arbitrary n) Clausen–Mathew–Naumann–Noel and Land–Mathew–Meier-Tamme show that when R is an $\mathbb{E}_1$-ring and $L_{T(n-1)\oplus T(n)}R=0$, then $L_{T(n)}\,\mathsf{K}(R)=0$.

2.5. **Morava K-theory non-vanishing and LQC at arbitrary heights.** Another reason we like algebraic K-theory is that is has a universal property. In particular, it is the universal additive invariant: for any group-like, additive functor [CDH+20, Def. 1.5.8, Def. 2.7.1]
$$E\colon\ \mathsf{Cat}_\infty^{\mathrm{ex}}\to \mathrm{S}$$
equipped with a natural transformation $\operatorname{Core}(-)\implies E(-)$, then there exists a natural transformation $\mathsf{K}\implies E$ sitting in the diagram
$$\begin{array}{ccc}\operatorname{Core}(-) & \Longrightarrow & \mathsf{K}(-)\\ & \searrow\!\!\searrow & \Downarrow\\ & & E(-)\end{array}$$
by [BGT13, CDH+20]. In particular, we get trace maps
$$\mathsf{K}\implies \operatorname{TC}\implies \operatorname{TC}^-\implies \operatorname{THH}.$$
When R is an E_∞-ring spectrum, we can also let $\mathsf{K}^{(n)}(R)=\mathsf{K}(\dots K(R)\dots)$ and similarly consider
$$\mathsf{K}^{(n)}(R)\to(\mathbb{T}^n\otimes R)^{h\mathbb{T}^n}.$$
The idea is that even though $\mathsf{K}^{(n)}(R)$ is rarely complex oriented, we can sometimes show that $(\mathbb{T}^n\otimes R/\operatorname{MU})^{h\mathbb{T}^n}$ is an even complex oriented ring spectrum or in the case of an E_∞ ring map $A\to B$ where B is an even complex oriented ring spectrum, we can map to $B^{h\mathbb{T}^n}$ and detect higher chromatic height information there.

Example 2.13. Again we list known examples
(1) (case n=-1) [Gro57] $\mathbb{Q}\otimes\mathsf{K}(\mathbb{F}_p)\neq 0$
(2) (case n=0) [Voe11] $K(1)_*\,\mathsf{K}(\mathcal{O}_F[1/p])\neq 0$
(3) (case n=1) [Aus10, AR02, BM08] $R=\ell,ku,k(1),\ell,KU,K(1)$, then $K(m)_*\,\mathsf{K}(R)\neq 0$ for $0\leq m\leq 2$.
(4) (arbitrary height) [HW20] When $0\leq m\leq n+1$, then
$$L_{K(m)}\,\mathsf{K}(BP\langle n\rangle)\neq 0$$
for $0\leq m\leq n+1$.
(5) (arbitrary height) [Yua21] When F is a field with $1/p\in F$
$$L_{K(m)}\,\mathsf{K}^{(n)}(F)\neq 0$$
for $0\leq m\leq n$. When E^{tC_p} is $K(n)$-local, then
$$L_{K(n+1)}\,\mathsf{K}(E^{tC_p})\neq 0.$$
(6) (height 2) [Ang18] A related result of a different flavor, is the result that
$$\mathsf{K}(\mathsf{K}(\mathbb{Z}))$$
detects a family of β-elements in $\pi_*\mathbb{S}$ at $p\geq 5$. This was proven by detecting these elements in $\operatorname{TC}_*^-(j)$ where $j=\tau_{\geq 0}L_{K(1)}\mathbb{S}$.

Problem 2.14. Can we draw explicit connections between $\mathsf{K}^{(2)}(\mathcal{O}_F)$ and modular forms over $\mathcal{O}_F$? What if we replace iterated algebraic K-theory with secondary K-theory?

Problem 2.15. Can we prove similar results for other families of $\mathbb{E}_n$-rings?

2.6. **Galois descent.** We say a map $A \to B$ of $\mathbb{E}_\infty$ rings is an E G-Galois extension if G acts on B by A-algebra maps and

(1) the map $A \to B^{hG}$ is a E-local equivalence, and
(2) the map $B \wedge_A B \to F(G_+, B)$ is a E-local equivalence.

Inspired by the Galois descent conjecture of Lichtenbaum, Ausoni–Rognes conjectured.

Conjecture 2.16 (Ausoni–Rognes Galois descent conjecture)**.** *Suppose $A \to B$ is a $K(n)$-local G-Galois extension, then*

$$T(n+1) \wedge \mathsf{K}(A) \to T(n+1) \wedge \mathsf{K}(A)^{hG}$$

is $K(n)$-local equivalence.

Example 2.17. We again list some examples:

(1) (Case $n = 0$) Galois descent conjecture of Lichtenbaum.
(2) (Case $n = 1$) For $L_p \to KU_p$, by [Aus10, AR02, BM08].
(3) (Arbitrary heights) When $A \to B$ is a G Galois extension for a finite group G such that the image of the transfer $\mathsf{K}_0(B) \otimes \mathbb{Q} \longrightarrow \mathsf{K}_0(A) \otimes \mathbb{Q}$ contains the unit, then

$$L_n^f \mathsf{K}(A) \to L_n^f \mathsf{K}(B)^{hG}$$

is an equivalence by [CMNN20]. For example the C_2 Galois extension $KO \to KU$, the $GL_2(\mathbb{Z}/n)$ Galois extension $\mathrm{TMF}[1/n] \to \mathrm{TMF}(n)$, $E(G,k)^{hH} \to E(G,k)$ is an H Galois extension where $E(G,k)$ is higher real K-theory associated to a formal group G over a perfect field k of characteristic p and H is a finite subgroup of the automorphisms $\mathrm{Aut}(G,k)$ of k.

Problem 2.18. Does algebraic K-theory satisfy *pro-finite* Galois descent? For example, is there a spectral sequence

$$\mathrm{H}^*(\mathbb{G}_n; T(n+1)_* \mathsf{K}(E_n)) \implies T(n+1)_* \mathsf{K}(L_{K(n)}\mathbb{S})?$$

Can condensed/pycknotic mathematics of Clausen–Scholze/Barwick–Haine be used here?

3. Stable diffeomorphisms of manifolds

We also care about algebraic K-theory because of applications to geometry. In particular, for a smooth manifold M,

$$\mathsf{K}(\mathbb{S}[\Omega M]) = \mathbb{S}[M] \oplus \mathrm{Wh}^{\mathrm{DIFF}}(M)$$

where $\mathrm{Wh}^{\mathrm{DIFF}}(M)$ encodes stable diffeomorphisms of M. This is already interesting when M is a point. Waldhausen [Wal84] suggested a program for computing algebraic K-theory of of the sphere spectrum by working up the tower

$$\mathsf{K}(\mathbb{S}_{(p)}) \longrightarrow \cdots \longrightarrow \mathsf{K}(L_2^f \mathbb{S}_{(p)}) \longrightarrow \mathsf{K}(L_1^f \mathbb{S}) \longrightarrow \mathsf{K}(H\mathbb{Q}).$$

By [MS93], the map

$$\mathsf{K}(\mathbb{S}_{(p)}) \to \operatorname*{holim}_n \mathsf{K}(\tau_{\geq 0} L_n^f \mathbb{S})$$

is an equivalence, an analogue of the chromatic convergence theorem. We also know $\mathsf{K}_*(\mathbb{S})$ in a range by work of Rognes [Rog03] and Blumberg–Mandell [BM19], but we still know less about $\mathsf{K}_*(\mathbb{S})$ then $\pi_*\mathbb{S}$ (and in fact $\pi_*\mathbb{S}$ is a summand).

Another approach to algebraic K-theory of the sphere spectrum was suggested by Dundas–Rognes [DR18]. Consider the Amitsur complex

$$\mathbb{S} \longrightarrow \mathrm{MU} \rightrightarrows \mathrm{MU}^{\otimes 2} \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots,$$

and apply algebraic K-theory. Then [DR18], prove that algebraic K-theory satisfies co-simplicial descent so that

$$\mathsf{K}(\mathbb{S}) \simeq \mathrm{Tot}(\mathsf{K}(\mathrm{MU}^{\otimes \bullet})).$$

Alternatively, when $A \to B$ is a G Galois extension for a finite group, so that $B \wedge_A B \simeq F(G_+, B)$, we produces a spectral sequence

$$H^*(G; \mathsf{K}_*(B))) \implies \mathsf{K}_*(B)^{hG}$$

and one can combine the results of Dundas–Rognes [DR18] with the results of Clausen–Mathew–Naumann–Noel [CMNN20] to compute $\mathsf{K}_*(A)$ in some cases.

Problem 3.1. Compute $\mathsf{K}_*(\mathrm{MU})$ and $\mathsf{K}_*(L_n^f\mathbb{S})$. Use Galois descent for a G Galois extension $A \to B$ to compute algebraic K-theory of A from algebraic K-theory of B, for example compute $F(1)_* \mathsf{K}(KO)$ by computing $F(1)_* \mathsf{K}(KU)$ at $p = 2$ and using descent.

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