# PRO-OBJECTS, PRO-HOMOTOPY THEORY AND PRO-DESCENT

GABE ANGELINI-KNOLL

## 1. Pro-objects, morphisms of pro-objects, and universal properties

Pro-objects arise naturally in algebra when you want to complete something, for example consider the functor $\mathbb{N}^{\mathrm{op}} \to \mathrm{Ab}$ given by sending $n$ to $\mathbb{Z}/p^n\mathbb{Z}$, then $\lim_{\mathbb{N}^{\mathrm{op}}} \mathbb{Z}/p^n\mathbb{Z} = \hat{\mathbb{Z}}_p$. This example should be familiar to everyone. The category $\mathbb{N}^{\mathrm{op}}$ is an example of a cofiltered category since the partially ordered set $\mathbb{N}$ is filtered and the functor $\mathbb{N}^{\mathrm{op}} \to \mathrm{Ab}$ is an example of a pro-object in abelian groups.

**Definition 1.1.** A category $\mathcal{I}$ is *filtered* if 1) for all $a, b \in \mathcal{I}$ there exists a $c \in \mathcal{I}$ and morphisms $a \to c$ and $b \to c$ in $\mathcal{I}$ and 2) for all pairs of morphisms $f, g\colon a' \to b'$ there exists a morphism $h\colon b' \to c'$ such that $h \circ f = h \circ g$. A *pro-object in a category* $\mathcal{C}$, in the classical sense, is a functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ where $\mathcal{I}$ is a small filtered category. If $\mathcal{I}$ is filtered then we say $\mathcal{I}^{\mathrm{op}}$ is cofiltered.

Pro-objects form a category pro-$\mathcal{C}$ where maps of pro-objects are given by

$$\hom_{\text{pro-}\mathcal{C}}(\{X_i\}_{i\in\mathcal{I}}, \{Y_j\}_{j\in\mathcal{J}}) = \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \hom_{\mathcal{C}}(X_i, X_j).$$

I'll explain this in the infinity category setting later. For now, I just want to point out that objects in $\mathcal{C}$ themselves can be considered as pro-objects with $\mathcal{J} = *$ and thus there is always a corresponding functor $\operatorname{colim}_{i\in\mathcal{I}} \hom(X_i, -)\colon \mathcal{C} \to \mathrm{Set}$, called a pro-representable functor. The subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ spanned by the pro-representable functors is equivalent to the category of pro-objects in $\mathcal{C}$ and it can be shown that a functor is pro-representable if and only if it commutes with finite limits (in which case we say the functor is left exact)[1, Appendix Cor. 2.8]. This motivates the infinity categorical definition.

We will assume knowledge of the definition of an $\infty$-category in the sense of [4] and notions of limits and colimits in this setting. Recall that for a regular cardinal $\kappa$, $\mathrm{Ind}_\kappa(\mathcal{C})$ is the subcategory of all functors $\mathcal{C} \to \mathcal{S}$ corresponding (via the $\infty$-categorical Grothendieck construction) to right fibrations $\mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is $\kappa$-filtered. An $\infty$-category $\mathcal{I}$ is $\kappa$-filtered if for every $\kappa$-small simplicial set $K$ and map $f : K \to \mathcal{I}$ there exists an extension $K * \Delta^0 \to \mathcal{I}$. The reason we care about $\mathrm{Ind}_\kappa(\mathcal{C})$ is that it is closed under $\kappa$-filtered colimits and it is universal amongst categories with this property. There is a similar property for the category of pro-objects, which we will discuss. Note that when $\kappa = \omega$ (the regular cardinal of all finite ordinals) we omit it from the notation. An $\infty$-category $\mathcal{C}$ is $\kappa$-accessible if $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}_0)$ for some small category $\mathcal{C}_0$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is $\kappa$-accessible if both $\mathcal{C}$ and $\mathcal{D}$ are accessible and $F$ preserves $\kappa$-filtered colimits.

**Definition 1.2.** Let $\mathcal{C}$ be an accessible $\infty$-category that admits finite limits, then pro-$\mathcal{C}$ is defined as the full subcategory $\mathrm{Fun}(\mathcal{C}, S)^{\mathrm{op}}$ spanned by those functors which are accessible and left exact. There is a canonical isomorphism pro-$\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ when $\mathcal{C}$ is also essentially small.

By Yoneda there is an embedding $j\colon \mathcal{C} \to \text{pro-}\mathcal{C}^{\mathrm{op}}$ and we may regard objects in $\mathcal{C}$ as object in pro-$\mathcal{C}$ as before. The essential image of the Yoneda embedding consists of cocompact objects in pro-$\mathcal{C}$ because 1) pro-$\mathcal{C}$ is closed under small filtered limits and 2) limits in pro-$\mathcal{C}$ are computed pointwise. Let $\mathcal{C}^\kappa$ be the full subcategory of $\kappa$-compact objects; i.e objects $c$ in $\mathcal{C}$ such that $Map_{\mathcal{C}}(c, -)$ preserves $\kappa$-filtered colimits. If $V\colon \mathcal{C} \to \mathcal{S}$ is accessible, then there exists a $\kappa$ such that $V$ is the left Kan extension of $V|_{\mathcal{C}^\kappa} = V_0\colon \mathcal{C}^\kappa \to \mathcal{S}$ and $V_0$ is an object in $(\text{pro-}\mathcal{C}^\kappa)^{\mathrm{op}}$. The object $V_0$ in pro-$\mathcal{C}^\kappa$ can therefore be written as a filtered limit of objects represented by objects in the essential image of the Yoneda embedding. Given object $X, Y$ in pro-$\mathcal{C}$, let $\{j(C_\alpha)\}, \{j(D_\beta)\}$ be collections of representable objects such that $\lim_\alpha j(C_\alpha) = X$ and $\lim_\beta j(D_\beta) = Y$. Then

$$\begin{aligned} Map_{\text{pro-}\mathcal{C}}(X, Y) &\simeq Map_{\text{pro-}\mathcal{C}}(\lim_\alpha j(C_\alpha), \lim_\beta j(D_\beta)) \\ &\simeq \lim_\beta Map_{\text{pro-}\mathcal{C}}(\{j(C_\alpha)\}, j(D_\beta)) \\ &\simeq \lim_\beta \operatorname{colim}_\alpha Map_{\text{pro-}\mathcal{C}}(j(C_\alpha), j(D_\beta))^1 \\ &\simeq \lim_\beta \operatorname{colim}_\alpha Map_{\mathcal{C}}(C_\alpha, D_\beta)^2 \end{aligned}$$

The category pro-$\mathcal{C}$ has a nice universal property, which can be stated using the following slogan: pro-$\mathcal{C}$ is constructed from $\mathcal{C}$ by freely adjoining all small filtered limits.

**Proposition 1.3** (Proposition A.8.1.6 [5])**.** Suppose $\mathcal{C}$ is an accessible $\infty$-category admitting finite limits and $\mathcal{D}$ is an $\infty$ category admitting all small filtered limits. Write $\mathrm{Fun}'(\text{pro-}\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by the functors pro-$\mathcal{C} \to \mathcal{D}$ that preserve small filtered limits. Then there is an equivalence of $\infty$-categories

$$\mathrm{Fun}'(\text{pro-}\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

given by composition with the Yoneda embedding.

*Quick sketch of proof:* Let $\mathcal{E}$ the smallest subcategory of $\mathrm{Fun}(\mathcal{C}, \hat{\mathcal{S}})^{\mathrm{op}}$ that contains the image of the Yoneda embedding and is closed under small filtered limits (where $\hat{\mathcal{S}}$ is the $\infty$-category of not necessarily small spaces). As before, write $\mathrm{Fun}'(\mathcal{E}, \mathcal{D})$ for the subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{D})$ spanned by small filtered limit preserving functors. Then 1) Show that $\mathrm{Fun}'(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ given by the Yoneda embedding is an equivalence of $\infty$-categories and 2) Show that there is an equivalence of categories $\mathcal{E} \simeq \text{pro-}\mathcal{C}$. □

As a consequence given a functor $f\colon \mathcal{C} \to \mathcal{D}$, then composing with the Yoneda embedding $\mathcal{C} \to \mathcal{D} \to \text{pro-}\mathcal{D}$ produces a functor pro(f): pro-$\mathcal{C} \to$ pro-$\mathcal{D}$ which commutes with small filtered limits. When $f$ is accessible and left exact, then this functor has a left adjoint $F\colon$ pro-$\mathcal{D} \to$ pro-$\mathcal{C}$ given by composing with $f$. Also, note that the Yoneda embedding $j\colon \mathcal{C} \to$ pro-$\mathcal{C}$ has right adjoint given sending $\{X_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ to $\lim_{i \in \mathcal{I}^{\mathrm{op}}} X_i$.

## 2. Pro-homotopy theory

Now we will specialize to when $\mathcal{C}$ is the $\infty$-category of spaces $\mathcal{S}$, the $\infty$-category of pointed connected spaces $\mathcal{S}^0_*$, or the $\infty$-category of spectra Sp. In this case, we have functors $\pi_n\colon \mathcal{S}^0_* \to \mathrm{Gp}$ and $\pi_n\colon \mathrm{Sp} \to \mathrm{Ab}$ which induce functors $\pi_i\colon$ pro-$\mathcal{S} \to$ pro-Gp and $\pi_i\colon$ pro-Sp $\to$ pro-Ab. We'd like to discuss the notion of equivalences of objects in pro-$\mathcal{S}$, pro-$\mathcal{S}^0_*$ and pro-Sp. First, note that Kerz-Strunk-Tamme [3] write "lim"$X_i$ for a an object in pro-$\mathcal{C}$ where $X_i \in \mathcal{C}$ for $i \in \mathcal{I}^{\mathrm{op}}$. If $X =$ "lim"$X_i$ is an object in pro-$\mathcal{S}^0_*$ or pro-Sp then we will simply write $\pi_n X$ and $H_n(X)$ for "lim"$\pi_n X_i$ and "lim"$H_n X_i$ as objects in pro-groups. For this discussion, we will think of functors $\mathcal{C} \to \mathcal{S}$ in pro-$\mathcal{C}$ via the corresponding functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ where $\mathcal{I}^{\mathrm{op}}$ is cofiltered. Maps in pro-$\mathcal{C}$ that are given by natural transformations of diagrams are called level maps (note that up to isomorphism all maps are level maps by Artin-Mazur [1, Appendix 3.2] ). We will restrict to some special cases that are sufficient for Kerz-Strunk-Tamme [3]. For a rigorous and general treatment of equivalences of pro-simplicial sets see [2].

**Definition 2.1.** A level map $f : X \to Y$ in pro-$\mathcal{S}$ is an equivalence if and only if the induced level map $f_*\colon \pi_n X \to \pi_n Y$ is an isomorphism pro-Gp. A level map $f : M \to N$ of pro-group-like-monoids in pro-$\mathcal{S}^0_*$ is an equivalence if and only if $f_*\colon \pi_n X \to \pi_n Y$ is an isomorphism of pro-groups for all $n \geq 0$.

[1] Holds because $j(D_\beta)$ is compact in $(\text{pro-}\mathcal{C})^{\mathrm{op}}$

[2] Holds by the Yoneda lemma.

**Remark 2.2.** Note that a level map of pro-groups $f : A \to B$ is an isomorphism if and only if for for all $s \in \mathcal{I}^{\mathrm{op}}$ there exists a $t \geq s$ and a commuting diagram

$$\begin{array}{ccc} A_t & \longrightarrow & A_t \\ \downarrow & \swarrow & \downarrow \\ A_s & \longrightarrow & A_t \end{array}$$

**Example 2.3.** A map of pro-simplicial rings $f : R \to S$ is an equivalence if and only if $f_* : \pi_n R \to \pi_n S$ is an isomorphism in pro-Gp for all $n \geq 0$.

**Definition 2.4.** A level morphism $f\colon\ X \to Y$ in pro-Sp is an equivalence if $f_*\colon\ \pi_n \tau_{\geq 0} X \to \pi_n \tau_{\geq 0} Y$ is an isomorphism in pro-Ab for all $n \geq 0$. An object in pro-Sp is contractible if it is equivalent to the trivial pro-spectrum. A diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & Y \end{array} \tag{1}$$

in pro-Sp of commuting level morphism is cartesian if it is a levelwise homotopy pullback in Sp.

As you would expect, the diagram (1) is cartesian if and only if $\mathrm{fib}(f) \simeq \mathrm{fib}(g)$ because fib is computed levelwise.

## 3. Pro-excision and pro-descent

For the rest of the talk all rings will be commutative. The reference for this section is [6] and I will use some of the language used there. Given a map of rings $f : R \to S$ and an ideal $I$ in $R$ we will say that $(f : R \to S, I)$ is an *excision situation* if $f(I)$ is an ideal in $S$ and $\ker f \cap I = 0$. We often write $I$ for $f(I)$. The two main examples are 1) extensions of rings $A \subset B$ with $I$ an ideal in $B$ that is contained in $A$ and 2) surjections of rings $R \twoheadrightarrow R/J$ and an ideal $I$ in $R$ such that $I \cap J = 0$. Any excision situation can be factored into $R \twoheadrightarrow f(I) \subset S$ so it suffices to treat these cases. If $(f : R \to S, I)$ is an excision situation then the square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & S/I \end{array}$$

is called a *Milnor square*. Also, if $(f : R \to S, I)$ is an excision situation then $(f : R \to S, I^s)$ is an excision situation for all $s \geq 1$. We will write $K(R)$ for the *non-connective* algebraic K-theory a ring $R$, which I will leave as a black box for this talk, but it takes values is spectra. Then if $(f : R \to S, I)$ is an excision situation, we define

$$K(R, I) := \mathrm{fib}(K(f) : K(R) \to K(S)))$$

and there is an induced map $K(R, I) \to K(S, I)$ so we can define

$$K(R, S, I) = \mathrm{fib}(K(R, I) \to K(S, I)).$$

If $(f : R \to S, I)$ is an excision situation, then Bass 1968 proved that $K_n(R, I) \cong K_n(S, I)$ for $n \leq 0$, Milnor and Swan 1971 showed that $K_1(R, I) \twoheadrightarrow K_1(S, I)$ is surjective and Geller-Weibel 1983 showed that $K_1(R, S, I) \cong \Omega^1_{S/R} \otimes_S I/I^2$. Consequently, the pro-abelian group $\{K_n(R, S, I^s)\}$ vanishes for all $n \leq 1$. (Note that there are examples where $K_1(R, S, I) \neq 0$ so the pro-excision statement is necessary, for example $\mathbb{Z}[\zeta_p] \subset \mathbb{Z}[\zeta_p] + p\mathbb{Z}[\zeta_p] \subset p\mathbb{Z}[\zeta_p]$ due to Swan 1971.) A consequence of the work of Kerz-Strunk-Tamme [3] is an improvement on these results:

**Corollary 3.1** (Corollary 4.12 [3])**.** For any excision situation $(f : R \to S, I)$ where $R$ and $S$ are Noetherian, the pro-spectrum $\{K(R, S, I^s)\}$ is contractible.

More generally, consider an abstract blow-up square $\Sigma$

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X; \end{array}$$

a diagram of schemes where if $X' \to X$ is proper, $Y \to X$ is a closed immersion, and the induced morphism $X'\ Y' \to X\ Y$ is an isomorphism. Then $Y' \to X'$ is also a closed immersion and we can associate an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ to $Y \to X$ (respectively, we can associate an ideal sheaf $\mathcal{J} \subset \mathcal{O}_{\tilde{X}}$ to $E \to \tilde{X}$). Then the $s-1$-st infinitesimal thickening of $Y$ in $X$ is denoted $Y_s$ and it corresponds to $\mathcal{I}^s$ (resp. the $Y'_s$ corresponds to $\mathcal{J}^s$).

Let Sch be the category of Noetherian schemes. Given a functor $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sp}$ and an abstract blow-up square $\Sigma$ of Noetherian schemes we will write $F(X, Y_s) = \mathrm{fib}(F(X) \to F(Y_s))$, $F(\tilde{X}, E_s) = \mathrm{fib}(F(\tilde{X}) \to F(E_s))$, and $F(X, \tilde{X}, Y_s, E_s)$ for the fiber of the induced map $F(X, Y_s) \to F(\tilde{X}, E_s)$. Then we say $F$ satisfies pro-cdh descent if the pro-object $\{F(X, \tilde{X}, Y_s, E_s)\}$ is contractible.

**Theorem 3.2** (Theorem A [3])**.** Algebraic K-theory satisfies pro-cdh descent.

This is one of the key results that is used to prove Weibel's conjecture.

## References


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