Periodicity in iterated algebraic K-theory

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MSU Topology Seminar
November 10, 2016

Outline

1. Classical forms of periodicity
2. Algebraic K-theory
3. Iterated algebraic K-theory
4. Periodicity in iterated algebraic K-theory of finite fields

Bott periodicity

The orthogonal groups $O(n)$ fit into a sequence

$$O(1) \subset O(2) \subset \cdots \subset \bigcup_i O(i) = O$$

and I will write O for the infinite union. Define ΩO to be mapping space of basepoint preserving maps from S^1 to O.

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Theorem (R. Bott)

There is a homotopy equivalence $\Omega^8 O \simeq O$, and as a consequence

$$\pi_{8k+j}(O) \cong \pi_j(O)$$

for all $k \geq 0$ and $0 \leq j \leq 7$.

Here $\pi_n(X) := [S^n, X]$ is defined to be homotopy classes of continuous basepoint preserving maps S^n to X.

Homotopy groups of the infinite orthogonal group

This is the first type of periodicity that arose in algebraic topology.

k	0	1	2	3	4	5	6	7	8k +j
$\pi_k(O)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\ldots$

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Question

Can this be used to pick out periodic families in the homotopy groups of spheres?

Image of J

The group $O(n)$ acts on $\mathbb{R}^n$, so it can be extended to a map

$$\varphi : O(n) \wedge S^n \to S^n$$

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The group $O(n)$ acts on $\mathbb{R}^n$, so it can be extended to a map

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by letting $O(n)$ fix the point at infinity. We can then map an element $f \in \pi_k(O(n))$ represented by

$$f : S^k \to O(n)$$

to the element in $\pi_{k+n}(S^n)$ represented by

$$S^k \wedge S^n \xrightarrow{f \wedge \mathrm{id}} O(n) \wedge S^n \xrightarrow{\varphi} S^n.$$

This produces a map

$$\pi_k(O(n)) \to \pi_{k+n}(S^n).$$

Image of J

If we take the colimit over n, then we produce a map

$$J : \operatorname*{colim}_{n\to\infty} \pi_k(O(n)) = \pi_k(O) \to \operatorname*{colim}_{n\to\infty} \pi_{n+k}(S^n) = \pi_k^s(S^0)$$

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and I will write $(\mathrm{Im}J)_k$ for the image of this map in degree k. In the 1960's, J.F. Adams studied this map and he proved that its image is the following:

k mod 8	1	2	3=4n-1	4	5	6	7= 4n-1
$(\mathrm{Im}J)_k$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/m(2n)\mathbb{Z}$	0	0	0	$\mathbb{Z}/2^{\epsilon}m(2n)\mathbb{Z}$

where $\epsilon = 0$ or 1.

Bernouli numbers

Consider the Taylor expansion of the function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \beta_k \frac{x^k}{k!}$$

then

$$\beta_{2s} = (-1)^{s-1} B_{2s}$$

where B_{2s} are the classical Bernouli numbers. Then $m(2n)$ is the denominator of $\beta_{2n}/4n$ when it is expressed in lowest terms.

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Example:

The image of J in degree 3 is the denominator of $\frac{B_2}{4} = \frac{1}{6} \cdot \frac{1}{4}$ and in fact $\pi_3^s(S^0) \cong \mathbb{Z}/24\mathbb{Z}$.

Image of J at odd primes

At odd primes p, the image of the map

$$J : \pi_k(O) \to \pi_n^s(S^0) \to \pi_n^s(\widehat{S^0}_p)$$

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denoted $(\mathrm{Im} J)_k$ can be realized by the homotopy groups of the p-completion of an infinite loop space denoted j. In particular, there is a multiplicative map

$$\widehat{S^0}_p \to \hat{j}_p$$

that induces an isomorphism $(\mathrm{Im} J)_k \cong \pi_k(\hat{j}_p)$ for $k > 0$ and an isomorphism $\pi_0(\widehat{S^0}_p) \cong \pi_0(\hat{j}_p)$.

Image of J at odd primes

At primes $p \geq 3$, the homotopy groups of $\hat{j}_p$ are

$$\begin{aligned} \pi_0(\hat{j}_p) &\cong \hat{\mathbb{Z}}_p \\ \pi_{(2p-2)k-1}(\hat{j}_p) &\cong \mathbb{Z}/p^{\nu_p(k)+1}\mathbb{Z} \\ \pi_i(\hat{j}_p) &\cong 0 \text{ otherwise} \end{aligned}$$

where $\nu_p(k)$ is the p-adic valuation of k. For example, when $k = 1$ the p-adic valuation is zero and we get $\mathbb{Z}/p\mathbb{Z}$ in degree $2p-3$, which is the first p-torsion in the homotopy groups of spheres

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such that no composite of it with itself is null homotopic. We can include the sphere in the bottom cell, then compose this map with itself some number of times, and then project to the top cell to produce the composite map

$$\Sigma^{(2p-2)k} S^m \longrightarrow \Sigma^{(2p-2)k} S^m/p \longrightarrow \dots \longrightarrow S^m/p \longrightarrow \Sigma^1 S^m.$$

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One can show that this composite map is not null homotopic and produces the element α_k. This method can generalized to define higher Greek letter elements.

β-family

We define $V(1)$ to be the cofiber of the map v_1

$$\Sigma^{2p-2} S^m/p \to S^m/p \to V(1).$$

When $p \geq 5$, $V(1)$ will have a periodic self map v_2

$$\Sigma^{2p^2-2} V(1) \to V(1)$$

and we can define the element β_k in $\pi_*^s(S^0)$ to be

$$\Sigma^{(2p^2-2)k} S^m \longrightarrow \Sigma^{(2p^2-2)k} V(1) \longrightarrow \ldots \longrightarrow V(1) \longrightarrow \Sigma^{2p} S^m,$$

which is a non null-homotopic map due to work of Miller, Ravenel, and Wilson.

Chromatic homotopy theory

There are (reduced) homology theories $K(n)_*(-)$ for each $n \geq 0$, with coefficients $K(n)_*(S^0) \cong \mathbb{F}_p[v_n^{\pm 1}]$ for $n > 0$ and $K(0)_*(X) \cong \widetilde{H}^*(X; \mathbb{Q})$.

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Definition

We say a finite p-local CW complex V has type n if $K(n)_* V \ncong 0$, but $K(i)_* V \cong 0$ for $i < n$.

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Theorem (Hopkins-Smith)

Every type n p-local CW complex V admits a v_n self map; i.e. a periodic map

$$\Sigma^d V \to V$$

that induces an isomorphism in $K(n)_*(-)$ and 0 in $K(i)_*(-)$ for $i \neq n$.

Greek letter elements

Example:

The p-localization of the sphere S^k is type 0, the mod p Moore spectrum S^k/p is type 1 and $V(1)$ is type 2.

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Given a type n p-local finite CW complex, one can (potentially) construct a periodic family in the homotopy groups of spheres corresponding to n-th Greek letter as we did for α and β. We will return to this later.

Algebraic K-theory

In the 1970's, Quillen defined higher algebraic K-theory extending the known low degree K-groups of rings. In particular, $K_0(R)$ is related to the class group, $K_1(R)$ is related to the group of units in R and $K_2(R)$ is related to the Brauer group.

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Definition (algebraic K-theory of a ring)

The degree 0 K-group of a ring is

$$K_0(R) = \mathrm{Gr}(\text{f.g proj. } R-\mathrm{Mod}, \oplus)$$

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Here $GL(R)$ is the infinite union $\bigcup GL_n(R)$, B is the classifying space construction and $+$ is H-spacification.

Quillen's computation K-theory of finite fields

Quilen then showed that there is an equivalence of fiber sequences

$$\begin{tikzcd}
\widehat{BGL(\mathbb{F}_q)^+}_p \arrow[r] \arrow[d, "\simeq"] & \widehat{BGL(\bar{\mathbb{F}}_q)^+}_p \arrow[r, "1-\mathrm{Frob}_q"] \arrow[d, "\simeq"] & \widehat{BGL(\bar{\mathbb{F}}_q)^+}_p \arrow[d, "\simeq"] \\
\widehat{F(\psi_q)}_p \arrow[r] & \widehat{BU}_p \arrow[r, "1-\psi_q"] & \widehat{BU}_p
\end{tikzcd}$$

where q is a topological generator of $\mathbb{Z}_p^\times$, ψ_q is the q-th Adams operation, Frob_q is the map induced by the Frobenius operator on $\bar{\mathbb{F}}_q$, and $\bar{\mathbb{F}}_q$ is the algebraic closure of the finite field $\mathbb{F}_q$.

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$$\pi_{2k-1}\widehat{K(\mathbb{F}_q)}_p \cong \hat{\mathbb{Z}}_p/(1-q^k)\hat{\mathbb{Z}}_p.$$

$$\pi_{2k}\widehat{K(\mathbb{F}_q)}_p \cong 0 \text{ if } k > 0$$

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We observe that this produces an isomorphism

$$\pi_k(\widehat{K(\mathbb{F}_q)}_p) \cong \pi_k(\hat{j}_p)$$

and in fact, this isomorphism can be realized by an equivalence

$$\widehat{K(\mathbb{F}_q)}_p \simeq \hat{j}_p.$$

The α-family detected in algebraic K-theory

Therefore, the map

$$S_p \to \widehat{K(\mathbb{F}_q)}_p$$

maps the α-family nontrivially into p-complete algebraic K-theory groups of $\mathbb{F}_q$, in other words the α-family, a periodic family of height one, is detected in the homotopy groups of $\widehat{K(\mathbb{F}_q)}_p$

Waldhausen algebraic K-theory

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Definition

Let R be a commutative ring spectrum, then

$$K(R) := K^{wald}(\text{fin. cell } R - \text{mod})$$

Iterated algebraic K-theory

In particular, this allows us to define iterated algebraic K-theory of a commutative ring. The main example I am interested in is $K(K(\mathbb{F}_q))$, which is defined as Waldhausen's algebraic K-theory of finite cell $K(\mathbb{F}_q)$-modules.

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Conjecture (Red-shift)

The n-th Greek letter family of the homotopy groups of spheres is detected in the n-th iterated algebraic K-theory of finite fields

$$K(\dots K(\mathbb{F}_q)\dots)$$

This is a modified version of the red-shift conjecture of Ausoni and Rognes.

Iterated algebraic K-theory of finite fields

Let $p \geq 5$ be a prime and let q be a topological generator of $\mathbb{Z}_p^\times$.

Theorem (A-K)

The β-family is detected in $\pi_*(K(K(\mathbb{F}_q))$; i.e. the unit map

$$\pi_*(\widehat{S^0}_p) \to \pi_*(K(\widehat{K(\mathbb{F}_q))}_p)$$

maps the β-family in the homotopy groups of spheres nontrivially.

Approximating iterated algebraic K-theory

This theorem is proven using trace methods. We approximate $K(K(\mathbb{F}_q))$ using the Bökstedt trace map

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Here THH may be thought of as a linear approximation to algebraic K-theory in the sense of Goodwillie calculus. One model for $THH(R)$ is $S^1 \otimes R$ and, therefore, topological Hochschild homology has a S^1 action. The shift in periodicity is visible in the homotopy S^1-fixed points of THH denoted

$$THH(K(\mathbb{F}_q))^{hS^1}.$$

Computations

Let q be a topological generator of $\hat{\mathbb{Z}}_p^\times$ where $p \geq 5$ be a prime number.

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Theorem (A-K)

There is an isomorphism

$$V(1)_*(THH(\widehat{K(\mathbb{F}_q)}_p) \cong P(\mu) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, \alpha_1, \lambda_1', \alpha_1\lambda_2, \lambda_1'\lambda_2, \alpha_1\lambda_1'\lambda_2\}.$$

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Theorem (A-K)

The classes v_2, β_1, and β_1' in $\pi_*(V(1))$ are detected in $V(1)_*(THH(K(\mathbb{F}_q))^{hS^1})$

These computations imply Theorem 4.1 using the multiplicative properties of the trace map.

Implications in number theory

One of the first examples of an L-function is the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Using analytic continuation, we can extend this function to be defined for negative integers and this produces special values

$$\zeta(-n) = -B_{n+1}/(n+1).$$

Notice that the denominator of $\zeta(-1-2n)$ is exactly the order of the image of J and also the order of $K_{2n-1}(\mathbb{F}_q)$.

Implications in number theory

More generally, the torsion in the orders of algebraic K-theory groups of rings of integers in number fields produces denominators of special values of Dedekind-Zeta functions.

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Question

Is there a relationship between iterated algebraic K-theory of finite fields and higher L-functions?

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At this point this is purely speculative, but my computations are the first step towards making this precise.

Thank You

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