Nilpotence and Stable Homotopy Theory II

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1 In the beginning there were CW complexes

Homotopy groups are such a natural thing to think about as algebraic topologists because they tell us about attaching maps of CW-complexes. If we have a map,

$$\Sigma^i S \longrightarrow S$$

and we take the cofiber we get a two cell complex and the map tells us about the structure of our two cell complex. If the map is null homotopic, then our cofiber splits as a wedge so it is natural to want to know when this map is null or when some itterate of it is null. If some itterate of this map is null we say it is nilpotent. If it is not nilpotent we say it is periodic. My favorite example of a periodic map of spheres that is not nilpotent is

$$S \xrightarrow{p} S \longrightarrow S/p$$

with cofiber the mod p Moore spectrum. In fact, the only periodic self maps of the sphere spectrum live in degree zero by Nashida. The nilpotence theorem discussed here is a generalization of this result and in fact this result is a special case.

Theorem 1.1. *(Nashida) Every element in $\pi_*(S)$ for $* > 0$ is nilpotent.*

I will now work in work in the category of spectra and restrict to finite and p-local spectra at times. Let me give a brief description of the objects in this category.

Definition 1.2. A spectrum is a sequence of spaces (nice enough to be weakly equivalent to CW-complexes) $\{E_i\}_{i\in\mathbb{Z}}$ with structure maps $\Sigma E_n \longrightarrow E_{n+1}$.

One model for what spectra should be is S-modules where S is the sphere spectrum. This is the approach in EKMM where smash products and many other useful constructions are defined. Now let me get to the most important definitions for my talk.

Definition 1.3.

1. A map $f : F \longrightarrow X$ is *smash nilpotent* if $f^{(n)} : F^{\wedge n} \longrightarrow X^{\wedge n}$ is null for some $n >> 0$.
2. A map $f : \Sigma^j F \longrightarrow F$ is *nilpotent* if $f^t : \Sigma^{jt} F \longrightarrow F$ is null for some $t >> 0$.
3. A map $f : S^n \longrightarrow R$ with R a ring spectrum is *nilpotent* if $f \in \pi_* R$ is nilpotent.

2 A New Hope

We would like to have some cohomology theory that detects these kind of maps. Ravenel conjectured that MU is the right cohomology to do the job. In 1988 Devinatz, Hopkins and Smith succeeded in proving that this is the case.

Theorem 2.1. *(DHS) In each of the situations above, f is nilpotent if F is finite and if $MU_*(f) = 0$.*

Remark. If the range of f is p-local, then f is nilpotent if $BP_*(f) = 0$

This is a great result and most of the 35 page paper is dedicated to proving a more general version of this theorem. But, we'd like to have a version of this theorem that is a little more computable. Recall that $\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$. We can use the Morava K-theories $K(n)$ to break this computation into computations at each prime and each height. Recall that $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}]$. It is built out of $MU_{(p)}$ by inverting v_n in the colimit and smashing this over $MU_{(p)}$ with a smash product of $M(t_k)$ for $k \neq p^n - 1$ where $M(t_k)$ is the cofiber of multplication by t_k on $MU_{(p)}$. Here I'm using the notatation $\pi_*(MU_{(p)}) = \mathbb{Z}_{(p)}[t_1, t_2, \cdots]$ where $|t_i| = 2i$ and $t_{p^n-1} = v_n$. We will now restrict to $p - local$ spectra, though as above we could have used any finite spectrum and then checked this at its p-localizations for each p.

3 Nilpotence

Theorem 3.1. *(Nilpotence theorem)*

1. *A map $f : S^n \longrightarrow R$ regarded as $f \in \pi_*(R)$ where R is a ring spectrum is nilpotent iff $K(n)_*(f)$ is nilpotent for all $0 \leq n \leq \infty$.*
2. *A self map $f : \Sigma^k F \longrightarrow F$ is nilpotent iff $K(n)_*(f)$ is nilpotent for all $0 \leq n < \infty$.*
3. *A map $f : F \longrightarrow X$ is nilpotent iff $K(n)_*(f) = 0$ for all $0 \leq n < \infty$*

We might ask if other cohomology theories detect nilpotence as well and how they are related to $K(n)$.

Definition 3.2. We say a ring spectrum E *detects nilpotence* if equivalently

1. For any ring spectrum R
$$\ker(\pi_*(R) \longrightarrow E_*(R))$$
consists of nilpotent elements.

2. A map $f : F \longrightarrow X$ is *smash nilpotent* if
$$1_E \wedge f : E \wedge F \longrightarrow E \wedge X$$
is null.

Corollary 3.3. *A cohomology theory E detects nilpotence iff*
$$K(n)_*(E) \neq 0$$
for all $0 \leq n \leq \infty$ *and all p.*

4 Thick Subcategories

If E is a $K(n)$ acyclic for any n and any prime p it doesn't detect nilpotence, so this leads us to consider the full subcategories of the category p-local finite specta with objects consisting of $K(n)$ acyclics. Denote $\mathcal{C}_0$ the category of p-local finite spectra and $\mathcal{C}_n$ the full subcategory of $K(n-1)$ acyclics. Then there is a filtration,
$$\mathcal{C}_\infty \subset \cdots \subset \mathcal{C}_{n+1} \subset \mathcal{C}_n \subset \cdots \mathcal{C}_0,$$
of the category of p-local finite spectra. The fact that $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ and that this is a proper containment is nontrivial.

Now we'd like to define a subcategory of spectra that is closed under some nice topological properties.

Definition 4.1. A full subcategory of the category of spectra is called *thick* if it is closed under cofibers, retracts, and weak equivalences.

Theorem 4.2. *(Thick Subcategory Theorem) Every thick subcategoy of* $\mathcal{C}_0$ *is of the form* $\mathcal{C}_n$ *for some n.*

If the full subcategory with objects having a property P form a thick subcategory of $\mathcal{C}_0$ then we say that property is generic.

5 Spanier-Whitehead Duality

A helpful technique used in the proof of the nilpotence theorem is to replace a map of the form $\Sigma^i F \longrightarrow F$ with the Spanier-Whitehead dual $S^i \longrightarrow D(F) \wedge F$. Where $D(F) = Map(F, S)$ where $Map(F, S)$ is the mapping spectrum from F to S. We could do the same for a map $F \longrightarrow X$ and replace it with the map $S^0 \longrightarrow D(F) \wedge X$. We can think of taking $D(F) \wedge F$ as in the case with vector spaces $V \oplus V^*$ and for finite spectra $D(F) \wedge F$ is a ring spectrum.

6 Bousfield Classes

Much of the development of chromatic stable homotopy theory depends on work of Bousfield on localization. We say that a spectrum X is in the Bousfield class of Y, denoted $< Y >$ if $X \wedge Y$ is nontrivial. One important result needed for the proof of the Nilpotence theorem is that $< BP >=< K(0) > \vee ... \vee < K(n) > \vee < P(n) >$. Where $\pi_*(P(n+1)) = \mathbb{F}_p[v_{n+1}, v_{n+2}, ...]$.

7 Proof of the Nilpotence theorem

Proof. First we show that 1. and 2. will follow from 3. To see that 1. follows from 3. we note that since $K(n)_*(\alpha)$ is the image of the Hurewicz homomorphism, which is a ring map, it must be nilpotent if α is nilpotent. For the other direction we look at the diagram

$$\begin{array}{ccc} S^{ktm} & \xrightarrow{(\alpha^t)^{(m)}} & R^{\wedge n} \\ & \searrow^{\alpha^{tm}} & \downarrow \mu \\ & & R \end{array}$$

and since $K(n)_*(\alpha)^t = K(n)_*(\alpha^t) = 0$, α^t is smash nilpotent by 3. and therefore α^{tm} is null homotopic for $m >> 0$, making α nilpotent.

2. follows from 1. by replacing $f : \Sigma^i F \longrightarrow F$ with $D(f) : S^i \longrightarrow D(F) \wedge F$ and recognizing that $R = D(F) \wedge F$ is a ring spectrum.

To prove 3. We replace the map $f : F \longrightarrow X$ with the map $D(f) : S^0 \longrightarrow D(F) \wedge X = E$. f is smash nilpotent if and only if $D(f)$ is smash nilpotent so we abuse notation and let $f := D(f)$. Suppose f is smash nilpotent. Then $1_{BP} \wedge f^{(m)}$ is nilpotent for $m >> 0$. Thus, $K(n)_*(f^{(m)}) = 0$ for all n by the Bousfield equivalence

$$< BP >=< K(0) > \vee ... \vee < K(n) > \vee P(n+1).$$

$K(n)_*(f^{(m)}) = 0$ iff $K(n)_*(f) = 0$ since $K(n)_*(f^{(m)}) \in K(n)_*(E^{\wedge m}) \cong \otimes_m K(n)_*(E)$ and it is identified with $f \otimes f \cdots \otimes f$ in $K(n)_*(E)$. Since $K(n)_*(E)$ is a free $K(n)_*$ module and $K(n)_*$ is a graded field (it has no zero divisors), $f = 0$ iff $f \otimes f \otimes \ldots f = 0$.

Conversely, suppose $K(n)_*(f) = 0$ for all n. Let

$$T = \text{hocolim}(S^0 \longrightarrow E \longrightarrow E \wedge E \longrightarrow \ldots).$$

Then $K(n) \wedge T \simeq *$ for all n. Since

$$< BP >=< K(0) > \vee \ldots \vee < K(n) \vee < P(n) >,$$

this implies $BP \wedge T \simeq *$. We can see this because $\varinjlim P(n) = H\mathbb{F}_p$. By Lemma 2.3 in HS, $BP \wedge T \simeq *$ iff $1_B P \wedge f^{(m)}$ is null homotopic for $m >> 0$. Thus, $BP_*(f^{(m)}) = 0$ so f is smash nilpotent. □

8 Smith-Toda complexes

I would like to give some examples of complexes that arise as cofibers of periodic maps. The simplest type of periodic map is a v_n map. I will explain what I mean by v_n map soon. First, let us consider the simplest case $v_0 = p$. We can realize the map geometrically and get a cofiber called $V(0)$, also known as the mod p Moore space

$$S \xrightarrow{p} S \longrightarrow S/p = V(0)$$

We don't have any heigher height v_n maps on the sphere spectrum by Nashida, but we do have a v_1 map on the S/p for p odd,

$$v_1 : \Sigma^{2(p-1)} S/p \longrightarrow S/p.$$

Adams studied this map in his paper on the image of the J homomorphism. This is the paper where he shows that in certain degrees the J homomorphism injects into the homotopy groups of spheres as a direct summand. This picked out a nice pattern in the homotopy groups of spheres coming from Bott periodicity in $\pi_*(SO(n))$. In this paper, he uses this map on the mod p Moore spectrum to pick out a family of elements in the homotopy groups of spheres called the α family. Now we might want to itterate this procedure and take v_2 map on the cofiber of the v_1 map,

$$v_2 : \Sigma^{2(p^2-1)} S/(p, v_1) \longrightarrow S/(p, v_1)$$

and this works, but only for primes $p \geq 5$. This produces the β family and assuming we could itterate this we would get the n-th greek letter family in the stable homotopy groups of spheres. These complexes were studied by Larry Smith and Hiroshi Toda in the early 70's in the hopes of generalizing what was known about the α family. It is known that these complexes don't exist for certain primes and that you really only have a power of a v_n map in certain cases (for example there is a v_1^4 map on $S/2$, but not a v_1 map). Lee Nave showed that $V((p+1)/2)$ doesn't exist for $p \geq 7$ for example.

9 Periodicity

Now by the nilpotence theorem nonnilpotent (or periodic) maps must be detected by some $K(n)$. A simple example is a v_n map (which will now refer to a v_n map and a v_n power map). It turns out that admitting a v_n map is a generic property.

Definition 9.1. Let $X \in C_0$ and $n > 0$. A self map $v : \Sigma^k X \longrightarrow X$ is a v_n self map if

$$K(m)_*(v) = \begin{cases} \text{an isomorphism} & \text{if } m = n \neq 0 \\ \text{nilpotent} & \text{if } m \neq n. \end{cases}$$

Theorem 9.2. *(Periodicity) $X \in \mathcal{C}_0$ admits a v_n self-map iff $X \in \mathcal{C}_n$. If x admits a v_n self-map, then for $N >> 0$, X admits a v_n self-map*

$$v : \Sigma^{p^N 2(p^n-1)} X \longrightarrow X$$

satisfying

$$K(m)_*(v) = \begin{cases} v_n^{p^N} & \text{if } m = n. \\ 0 & o.w. \end{cases}$$