

# COMMUTING UNBOUNDED HOMOTOPY LIMITS WITH MORAVA K-THEORY

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ABSTRACT. This paper provides conditions for Morava  $K$ -theory to commute with certain homotopy limits. These conditions extend previous work on this question by allowing for homotopy limits of sequences of spectra that are not uniformly bounded below. As an application, we prove the  $K(n)$ -local triviality (for sufficiently large  $n$ ) of the algebraic  $K$ -theory of algebras over truncated Brown-Peterson spectra, building on work of Bruner and Rognes and extending a classical theorem of Mitchell on  $K(n)$ -local triviality of the algebraic  $K$ -theory spectrum of the integers for large enough  $n$ .

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## 1. INTRODUCTION

Given a generalized homology theory  $E_*$  and a sequence

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$$

of spectra, one often needs to know, for computations, whether there is an isomorphism

$$\lim_i E_*(X_i) \cong E_*(\operatorname{holim}_i X_i).$$

This cannot be true in full generality. For example, the limit of the sequence

$$\cdots \rightarrow S/p^2 \rightarrow S/p$$

is the  $p$ -complete sphere  $\hat{S}_p$  and therefore

$$H_*(\operatorname{holim}_i S/p^i; \mathbb{Q}) \cong H_*(\hat{S}_p; \mathbb{Q}) \cong \mathbb{Q}_p$$

whereas  $H_*(S/p^i; \mathbb{Q}) \cong 0$  for  $i \geq 1$  and therefore  $\lim_i H_*(S/p^i; \mathbb{Q}) \cong 0$ , and  $\lim_i^1 H_*(S/p^i; \mathbb{Q})$  also vanishes, so we do not even have hope of recovering

$$H_*(\operatorname{holim}_i S/p^i; \mathbb{Q})$$

from a ‘‘Milnor sequence.’’

This motivates the question: what conditions on  $E_*$  and the sequence

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$$

allow us to commute the homotopy limit with  $E_*$ ? There are known results along these lines, most famously a commonly-used result of Adams from [1], but the usual hypothesis is that the spectra  $X_i$  are uniformly bounded below and the homology theory  $E_*$  is connective. In this paper, we remove each of these assumptions, under some reasonable additional hypotheses. Our particular focus is on the case where  $E_*$  is a Morava  $K$ -theory  $K(n)_*$ .<sup>1</sup>

This paper is written with a view towards filtered spectra that arise when studying topological periodic cyclic homology, in particular, the Greenlees filtration (21) on topological periodic cyclic homology

$$TP(R) := THH(R)^{t\mathbb{T}},$$

is not uniformly bounded below. Nevertheless, these filtered spectra often have nice enough homological properties to apply the main result of this paper.

Following the red-shift program of Ausoni and Rognes [6], we are most interested in the chromatic complexity of topological periodic cyclic homology. Therefore, a generalized homology theory of primary interest is Morava  $K$ -theory  $K(n)_*$ . Calculating Morava  $K$ -theory of topological periodic cyclic homology using the Greenlees filtration requires that one be able to commute a non-bounded-below generalized homology theory (Morava  $K$ -theory) with a non-uniformly-bounded-below homotopy limit, so existing results on generalized homology of limits, like Adams’ theorem from [1] reproduced as Theorem 3.2 below, do not suffice.

Classically, the vanishing of the Margolis homology  $H(H_*(X; \mathbb{F}_p), Q_n)$  of a bounded-below spectrum  $X$  with finite-type mod  $p$  homology implies that  $X$  has  $K(n)_*$ -acyclic. However, even if each spectrum  $X_i$  is bounded below, has finite-type homology, and  $H(H_*(X; \mathbb{F}_p), Q_n)$  vanishes, it is not always the case that  $\operatorname{holim}_i X_i$  has trivial  $K(n)$ -homology. Our main theorem establishes sufficient conditions for this homotopy limit to indeed have trivial  $K(n)$ -homology. Given a spectrum  $X$ , we adopt the notation that  $X^{<N}$  is its Postnikov truncation constructed by attaching cells to kill all the homotopy groups of  $X$  in degrees  $\geq N$ . Our main result may then be summarized as follows.

**Theorem 1.1.** *(Theorem 3.5) Suppose  $M$  is an integer and*

$$\cdots \rightarrow Y_2 \rightarrow Y_1 \rightarrow Y_0$$

*is a sequence of bounded below finite-type spectra which are  $H\mathbb{F}_p$ -nilpotently compete, such that the largest grading degree of a comodule primitive in  $H_*(Y_i; \mathbb{F}_p)$  is*

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<sup>1</sup>The easiest case of a Morava  $K$ -theory—in particular, the only case which is bounded below—is  $K(0)_*$ , which coincides with rational homology. So the above example with the rational homology of the  $p$ -complete sphere shows that some hypotheses are needed in order to commute Morava  $K$ -theory with a sequential limit.

strictly less than  $M$ , the homology groups  $H_*(Y_i; \mathbb{F}_p)$  and  $H_*(Y_i^{<M}; \mathbb{F}_p)$  are finitely generated for each  $i$ , and the limit

$$\varinjlim_{i \rightarrow \infty} H(H_*(Y_i; \mathbb{F}_p), Q_n)$$

of the Margolis homologies vanishes. Then the  $n$ -th Morava K-theory of  $\text{holim}_i Y_i$  is trivial. That is, we have an isomorphism

$$K(n)_*(\text{holim}_i Y_i) \cong 0.$$

This result relies on another result which resolves the question of when Postnikov truncations  $X \rightarrow X^{<N}$  induce injections in homology.

**Theorem 1.2.** (Theorem 2.6) *Let  $M, N$  be integers with  $N \geq M$ , and let  $X$  be a bounded below  $H\mathbb{F}_p$ -nilpotent spectrum with the grading degrees of the comodule primitives in  $H_*(X; \mathbb{F}_p)$  bounded above by  $M$ . Then the map*

$$H_*(X; \mathbb{F}_p) \rightarrow H_*(X^{<N}; \mathbb{F}_p),$$

*induced by the canonical map  $X \rightarrow X^{<N}$ , is injective.*

As the main application in the present paper, we prove a higher chromatic height analogue of Mitchell's theorem for algebraic K-theory of truncated Brown-Peterson spectra, building on work of Bruner and Rognes [11]. In particular, Mitchell proves in [26] that

$$K(m)_*(K(\mathbb{Z})) = 0$$

for  $m \geq 2$ , and consequently the same vanishing of Morava K-theory occurs for any  $H\mathbb{Z}$ -algebra. Let  $BP\langle n \rangle$  denote the  $p$ -completion of the truncated Brown-Peterson spectrum. As an application of the main result of this paper, we prove the following result, building on [11, Prop. 6.1].

**Theorem 1.3.** (Theorem 4.10) *There are isomorphisms*

$$K(m)_*(K(BP\langle n \rangle)) \cong 0$$

*for  $m \geq n + 2$ , which holds at all primes when  $n = 0, 1$  and holds at  $p = 2, 3$  when  $n = 2$ . Consequently, the same vanishing of Morava K-theory holds for any  $BP\langle n \rangle$ -algebra.<sup>2</sup>*

In particular, we note that this recovers the vanishing of  $K(m)_*(K(BP\langle 0 \rangle))$  for  $m \geq 2$  of [26] and the vanishing of  $K(m)_*(K(BP\langle 1 \rangle))$  for  $m \geq 3$  shown by Ausoni-Rognes [5], although our proof uses entirely different methods. After work of Mitchell in 1990 [26] and work of Ausoni-Rognes in 2002 [5], there has been renewed interest and progress on questions of this nature in the last five years. In particular, building on joint work with Naumann and Noel in [24], Mathew (unpublished) has shown that  $K(i)_*K(E_n)$  vanishes for  $i \geq n + 2$ , where  $E_n$  is the  $n$ -th Lubin-Tate  $E$ -theory. Also, Land, Meier, and Tamme [19] recently proved several results of this flavor, for example, they show that if  $R$  is a  $K(1)$ -acyclic  $E_\infty$ -algebra then  $K(n)_*(K(R)) = 0$  for  $n \geq 2$  (as a consequence of [19, Cor. 2.29]). We regard our Theorem 4.10 as complementary to these recent results of Mathew or of Land-Meier-Tamme: our results are not special cases of theirs, and theirs are not special cases of ours.

<sup>2</sup>This result also holds for arbitrary  $p$  and  $n$  if certain plausible-looking conditions on  $E_4$ -ring spectra can be shown to hold. See Remark 4.3 for explanation.

Our work in this paper was motivated initially by a conjecture in a first draft of a paper by the first author and Quigley [3]. We restate the version for topological periodic cyclic homology. Here we write  $TP(y(n))[k]$  for the Greenlees filtration of topological periodic cyclic homology of  $y(n)$  (see (21)).

**Conjecture 1.4.** There are isomorphisms

$$\lim_{k \rightarrow -\infty} K(m)_*(TP(y(n))[k]) \cong K(m)_*(TP(y(n)))$$

for  $1 \leq m \leq n$  and all  $1 \leq n < \infty$ .

This conjecture is resolved by the main theorem in the present paper, given above as Theorem 1.1, together with calculations of the first author and Quigley which verify that the hypotheses (on comodule primitives and finite-typeness of homology) of that main theorem are satisfied. The first author and Quigley plan to update [3] with these stronger results.

**1.1. Conventions.** When  $\kappa$  is a cardinal number and  $A$  an object in some category, we will write  $A^\kappa$  for the  $\kappa$ -fold categorical product and  $All^\kappa$  for the  $\kappa$ -fold categorical coproduct. If our category is additive, then we write  $A^{\oplus \kappa}$  for the  $\kappa$ -fold categorical biproduct of  $A$  with itself. Let  $R$  be a commutative ring. In this paper, the term “finite type” is used in the sense common in algebraic topology: a graded  $R$ -module  $V$  is *finite type* if, for each integer  $n$ , the grading degree  $n$  summand  $V^n$  is a finitely generated  $R$ -module. In particular, if  $R$  is a field, then we say  $V$  is finite type if, for each integer  $n$ , the grading degree  $n$  summand  $V^n$  is a finite dimensional vector space.

We will write  $P(x_1, \dots, x_n)$  for a polynomial algebra with generators  $x_1, \dots, x_n$  over  $\mathbb{F}_p$ ,  $E(x_1, \dots, x_n)$  for an exterior algebra over  $\mathbb{F}_p$  with generators  $x_1, \dots, x_n$  and we write  $P_k(x_1, \dots, x_n)$  for the truncated polynomial algebra with generators  $x_1, \dots, x_n$ , where  $P_k(x)$  is often also denoted  $\mathbb{F}_p[x]/x^k$  in the literature. Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , we will write  $\mathcal{D}^{\mathcal{C}}$  for the category of functors from  $\mathcal{C}$  to  $\mathcal{D}$ .

**1.2. Organization.** In Section 2, we give sufficient conditions for the canonical map  $X \rightarrow X^{<N}$  to induce an injection on homology. This is a key result used to prove the main theorem in Section 3 and we believe it is a useful contribution to the literature in its own right. In Section 3, we prove the main theorem about when we may commute Morava K-theory with a non-uniformly bounded below sequential limits. In Section 4, we give our main application, which is a proof of a higher chromatic height analogue of Mitchell’s theorem about vanishing of Morava K-theory of algebraic K-theory of the integers. Finally, we give some supplementary results and background on Margolis homology in Appendix A.

**1.3. Acknowledgements.** The first author would like to thank J.D. Quigley for discussions related to this project and B. Dundas for expressing interest in the results in this paper. This paper began as a batch of rough notes by the second author about the technical question of when one can commute a sequential homotopy limit with a smash product, in “bad” cases where the hypotheses of Adams’ theorem are not satisfied. The first author joined the paper later and since then several of the proofs in Section 3 have been significantly reworked. Also, Section 4 containing the applications to truncated Brown-Peterson spectra are a primary contribution of the second author leading to the paper’s present form. The second

author is therefore grateful to the first author and to J.D. Quigley for their patience in waiting for these results to be made publicly available.

## 2. WHEN DOES KILLING HOMOTOPY INDUCE AN INJECTION IN HOMOLOGY?

This section proves a general result, Theorem 2.6, which will be used to prove Theorem 3.5. However, it takes some work to prove and is useful in its own right. The question is when, given a spectrum  $X$  and an integer  $N$ , the map  $X \rightarrow X^{<N}$  given by attaching cells to kill all the homotopy groups of  $X$  in degrees  $\geq N$  induces an injection in mod  $p$  homology groups. Theorem 2.6 gives some practical sufficient conditions for  $H_*(X; \mathbb{F}_p) \rightarrow H_*(X^{<N}; \mathbb{F}_p)$  to be injective. A simple example where the hypotheses, and therefore the result, hold can be constructed by letting  $X = S^0$ ,  $M = N = 1$  so that  $\mathbb{F}_p \hookrightarrow (A//E(0))_*$  is an inclusion. A simple example where the hypotheses are *not* satisfied is the case where  $X = S^0$  and  $M = N = 0$ , where of course  $X^{<N}$  is the 0 spectrum and the result cannot hold.

The notation  $A_*$  in the statement of Lemma 2.1 refers to the dual mod  $p$  Steenrod algebra, as usual, but the lemma and its proof both work for any graded Hopf algebra over a field.

**Lemma 2.1.** *Let  $p$  be a prime number, let  $I$  be a set, and for each  $i \in I$ , let  $M_i$  be a graded  $A_*$ -comodule. Then, for each integer  $t$  and nonnegative integer  $s$ , the natural map of  $\mathbb{F}_p$ -vector spaces*

$$\coprod_{i \in I} \text{Cotor}_{A_*}^{s,t}(\mathbb{F}_p, M_i) \rightarrow \text{Cotor}_{A_*}^{s,t}\left(\mathbb{F}_p, \coprod_{i \in I} M_i\right)$$

is an isomorphism.

*Proof.* This result follows by observing that each step in the construction of the cobar complex of  $A_*$ , with coefficients in an  $A_*$ -comodule  $M$ , commutes with coproducts in the variable  $M$ .  $\square$

**Definition 2.2.** *Let  $p$  be a prime number and let  $M$  be an integer. We will say  $X$  satisfies condition  $H(M)$  if it is bounded below and  $H\mathbb{F}_p$ -nilpotently complete<sup>3</sup> and the  $A_*$ -comodule primitives of  $H_*(X; \mathbb{F}_p)$  are trivial in grading degrees  $\geq M$ .*

We suppress the prime  $p$  from the notation  $H(M)$  because it will always be clear from the context.

**Remark 2.3.** Given a bounded below  $H\mathbb{F}_p$ -nilpotent complete spectrum  $X$ , we note that the 0-line of the  $E_2$ -page of the Adams spectral sequence

$$\text{Ext}_{\text{Comod}(A_*)}^{*,*}(\mathbb{F}_p, H_*(X)) \rightarrow \pi_* X$$

is isomorphic to

$$\text{Hom}_{\text{Comod}(A_*)}(\mathbb{F}_p, H_* X),$$

which is isomorphic to the sub- $A_*$ -comodule of  $A_*$ -comodule primitives in  $H_* X$ . So, if  $X$  is bounded below and  $H\mathbb{F}_p$ -nilpotently complete, then condition  $H(M)$  is equivalent to the following statements being true for all integers  $j \geq M$ :

<sup>3</sup>Readers who are not used to thinking about nilpotent completion may be relieved to know that a bounded-below spectrum is  $H\mathbb{F}_p$ -nilpotently complete if and only if its homotopy groups are Ext- $p$ -complete, by [9, Prop. 2.5]. In particular, if all the homotopy groups of a bounded-below spectrum are  $p$ -adically complete, then that spectrum is also  $H\mathbb{F}_p$ -nilpotently complete.

(1) the Hurewicz map

$$\pi_j(X) \rightarrow H_j(X; \mathbb{F}_p)$$

is zero, and

(2) for all  $r \geq 2$ , the  $H\mathbb{F}_p$ -Adams spectral sequence for  $X$  does not have any nonzero  $d_r$ -differentials supported in bidegree  $(0, j)$  (i.e.,  $s = 0$  and  $t = j$ ).

It is the above two conditions that actually play a role in the proofs in this section, but the form of condition  $H(M)$  given in Definition 2.2 is a relatively familiar and easily-checked condition which is equivalent (for bounded below  $H\mathbb{F}_p$ -nilpotently complete spectra) to the two properties given above.

In Theorem 6.6 of [9], Bousfield proved that, if  $E$  is a connective ring spectrum with  $\pi_0(E) \cong \mathbb{F}_p$  and  $X$  is a connective spectrum, then the  $E$ -nilpotent completion  $\hat{X}_E$  of  $X$  is weakly equivalent to the Bousfield localization  $L_EX$ . As a special case, we have:

**Lemma 2.4.** *Let  $X$  be a connective spectrum. Then the  $H\mathbb{F}_p$ -nilpotent completion map  $X \rightarrow \hat{X}_{H\mathbb{F}_p}$  is an  $H\mathbb{F}_p$ -local equivalence. That is, the induced map of spectra  $X \wedge H\mathbb{F}_p \rightarrow \hat{X}_{H\mathbb{F}_p} \wedge H\mathbb{F}_p$  is a weak equivalence.*

**Lemma 2.5.** *Let  $p$  be a prime number and let  $I$  be a set. Let  $X$  satisfy condition  $H(M)$  for some integer  $M$ . For each  $i \in I$ , let  $f_i: S^n \rightarrow X$  be a map of spectra which induces the zero map in mod  $p$  homology, and let  $f: (\coprod_{i \in I} S^n)_{H\mathbb{F}_p} \rightarrow \hat{X}_{H\mathbb{F}_p} \xrightarrow{\cong} X$  denote the map given by  $H\mathbb{F}_p$ -nilpotent completion and the universal property of the coproduct applied to the set of maps  $\{f_i : i \in I\}$ . Suppose that the map*

$$(1) \quad \pi_n \left( \coprod_{i \in I} S^n \right) \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

is injective. Finally, suppose that  $n \leq M$ . Then the following statements are true:

- (1) the map  $H_*(X; \mathbb{F}_p) \rightarrow H_*(\text{cof } f; \mathbb{F}_p)$  is injective, and
- (2)  $\text{cof } f$  satisfies condition  $H(M)$ .

*Proof.* The  $H\mathbb{F}_p$ -nilpotent completion map  $\coprod_{I_k} S^n \rightarrow (\coprod_{I_k} S^n)_{H\mathbb{F}_p}$  is an  $H\mathbb{F}_p$ -local equivalence, by Lemma 2.4, so we have a natural isomorphism

$$\coprod_{I_k} H_*(S^n; \mathbb{F}_p) \xrightarrow{\cong} H_* \left( \left( \coprod_{I_k} S^n \right)_{H\mathbb{F}_p}; \mathbb{F}_p \right).$$

Since each  $f_i$  induces the zero map in mod  $p$  homology, so does the map  $\coprod_{i \in I_k} f_i$ , and consequently so does the composite

$$(2) \quad \coprod_{I_k} S^n \xrightarrow{\coprod_{i \in I_k} f_i} \coprod_{I_k} X \xrightarrow{\nabla} X,$$

i.e., the map  $\coprod_{I_k} S \rightarrow X$  given by the universal property of the coproduct. (The symbol  $\nabla$  in (2) stands for the fold map, given again by the universal property of

the coproduct.) We have the commutative square

$$\begin{array}{ccc}
 (\coprod_{I_k} S^n) \wedge H\mathbb{F}_p & \longrightarrow & X \wedge H\mathbb{F}_p \\
 \downarrow \cong & & \downarrow \cong \\
 (\coprod_{I_k} S^n)_{H\mathbb{F}_p} \wedge H\mathbb{F}_p^{f \wedge H\mathbb{F}_p} & \xrightarrow{\hat{f}} & \hat{X}_{H\mathbb{F}_p} \wedge H\mathbb{F}_p
 \end{array}$$

in which the vertical maps are isomorphisms in SHC by Lemma 2.4. The top horizontal map was already shown to be zero, so the bottom horizontal map is as well. So  $f$  induces the zero map in mod  $p$  homology. So we have the short exact sequence of  $A_*$ -comodules

$$(3) \quad 0 \rightarrow H_*(X; \mathbb{F}_p) \rightarrow H_*(\text{cof } f; \mathbb{F}_p) \rightarrow \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p) \rightarrow 0$$

and hence the claimed injectivity of  $H_*(X; \mathbb{F}_p) \rightarrow H_*(\text{cof } f; \mathbb{F}_p)$ , as well as an induced long exact sequence

$$(4) \quad \begin{array}{ccc}
 0 \longrightarrow & \text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, H_*(X; \mathbb{F}_p)) & \xrightarrow{a} \text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, H_*(\text{cof } f; \mathbb{F}_p)) \\
 & & \searrow b \\
 & \text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p)) & \longrightarrow \dots
 \end{array}$$

Let  $j \geq M$ . We want to show that the map

$$\pi_j(\text{cof } f) \rightarrow H_j(\text{cof } f; \mathbb{F}_p)$$

is zero. A nonzero element  $x \in H_j(\text{cof } f; \mathbb{F}_p)$  in the image of (2) corresponds to a nonzero element in  $\text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, H_*(\text{cof } f; \mathbb{F}_p))$ , and in the long exact sequence in  $\text{Cotor}_{A_*}^{*,*}(\mathbb{F}_p, -)$  induced by (3), such an element can arise in two ways: either from a nonzero element in  $\text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p))$  or from a nonzero element in  $\text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, H_*(X; \mathbb{F}_p))$ . We handle these possibilities separately:

**If  $b(x) = 0$ :** choose  $\tilde{x} \in \text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, H_*(X; \mathbb{F}_p))$  such that  $a(\tilde{x}) = x$ . Since  $X$  satisfies condition  $H(M)$ , we know that  $\tilde{x}$  does not survive the Adams spectral sequence for  $X$  (see Remark 2.3). Therefore, there must be a nonzero differential on  $\tilde{x}$ , contradicting the assumption on differentials in the Adams spectral sequence for  $X$ .

**If  $b(x) \neq 0$ :** then  $j = n + 1$ , since otherwise  $\text{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p))$  vanishes. We have  $\text{Cotor}_{A_*}^{0,n+1}(\mathbb{F}_p, \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p)) \cong \coprod_{i \in I} \mathbb{F}_p$  (using Lemma 2.1). Every nonzero element in bidegree  $(0, n + 1)$  of the  $E_2$ -term survives the Adams spectral sequence for  $\coprod_{i \in I} \Sigma S^n$ , including  $b(x)$ , so the image of  $x \in \pi_{n+1}(\text{cof } f)$  in  $\pi_{n+1}(\Sigma \coprod_{i \in I} S^n)$  is a nonzero element of Adams filtration zero. Consequently, we have a commutative diagram with exact

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$$\begin{array}{ccccc}
\pi_{n+1}(\operatorname{cof} f) & \longrightarrow & \pi_{n+1}(\Sigma \coprod_{i \in I} S^n) & \longrightarrow & \pi_{n+1}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{n+1}(\operatorname{cof} f) \otimes_{\mathbb{Z}} \mathbb{F}_p & \xrightarrow{0} & \pi_{n+1}\left(\Sigma \coprod_{i \in I} S^n\right) \otimes_{\mathbb{Z}} \mathbb{F}_p & \hookrightarrow & \pi_{n+1}(X) \otimes_{\mathbb{Z}} \mathbb{F}_p \\
\downarrow & & \downarrow & & \downarrow \\
H_{n+1}(\operatorname{cof} f; \mathbb{F}_p) & \longrightarrow & H_{n+1}\left(\Sigma \coprod_{i \in I} S^n; \mathbb{F}_p\right) & \longrightarrow & H_{n+1}(X; \mathbb{F}_p)
\end{array}$$

where the hooked arrow is a monomorphism by the assumption that (1) is injective. An easy diagram chase then gives that the image of  $x$  in  $\pi_{n+1}(\Sigma \coprod_{i \in I} S^n)$  must be divisible by  $p$ , hence its Hurewicz image in the group  $H_{n+1}(\Sigma \coprod_{i \in I} S^n; \mathbb{F}_p)$  is zero. That is, the image of  $x \in \pi_{n+1}(\operatorname{cof} f)$  in  $\pi_{n+1}(\Sigma \coprod_{i \in I} S^n)$  has positive Adams filtration, a contradiction. So  $b(x)$  cannot be nonzero.

So there cannot be a nonzero element  $x$  in the image of (2), as desired.

Continuing to assume  $j \geq M$ , we still need to prove that, for all  $r \geq 2$ , the Adams spectral sequence for  $\operatorname{cof} f$  does not have any nonzero  $d_r$ -differentials supported in bidegree  $(0, j)$ . Suppose, to the contrary, that  $r \geq 2$  and  $d_r(x) = y \neq 0$ , with  $x$  in bidegree  $(0, j)$  of the Adams spectral sequence for  $\operatorname{cof} f$ . We represent  $x$  by an element  $\bar{x}$  in bidegree  $(0, j)$  of the  $E_2$ -term, and again using exact sequence (4), there are two cases to consider:

- If  $b(\bar{x}) = 0$ :** then choose  $\tilde{x} \in \operatorname{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, H_*(X; \mathbb{F}_p))$  such that  $a(\tilde{x}) = \bar{x}$ . Either there exists an Adams differential on  $\tilde{x}$ , contradicting the assumption that  $X$  has no nonzero Adams differentials supported in bidegree  $(0, j)$ ; or  $\tilde{x}$  represents a nonzero element in the image of  $\pi_j(X) \rightarrow H_j(X; \mathbb{F}_p)$ , contradicting the hypothesis that  $X$  satisfies condition  $H(M)$  (see condition (1) in Remark 2.3).
- If  $b(\bar{x}) \neq 0$ :** then  $j = n + 1$  since otherwise  $\operatorname{Cotor}_{A_*}^{0,j}(\mathbb{F}_p, \coprod_{i \in I} H_*(\Sigma S^n, \mathbb{F}_p))$  vanishes. But now we have  $n + 1 = j \geq M$ , contradicting the assumption that  $n \leq M$ .

So  $x$  cannot exist. □

**Theorem 2.6.** *Let  $p$  be a prime number, let  $M, N$  be integers with  $N \geq M$ , and let  $X$  satisfy condition  $H(M)$ . Let  $X^{<N}$  be  $X$  with cells attached to kill all the homotopy groups of  $X$  in degrees  $\geq N$ . Then the map  $H_*(X; \mathbb{F}_p) \rightarrow H_*(X^{<N}; \mathbb{F}_p)$ , induced by the canonical map  $X \rightarrow X^{<N}$ , is injective.*

*Proof.* This is a proof by induction. Suppose  $k$  is an integer,  $k \geq N$ , and that we have already constructed a sequence of maps

$$X = X(N) \rightarrow X(N+1) \rightarrow \cdots \rightarrow X(k)$$

satisfying the properties:

- (1) for each  $j > N$ , the map  $X(j-1) \rightarrow X(j)$  induces an isomorphism in  $\pi_i$  for all  $i < j-1$ , and
- (2) for each  $j > N$ ,  $\pi_{j-1}(X(j))/p \cong 0$ , and



- (3) for each  $j > N$ , the induced map  $H_*(X(j-1); \mathbb{F}_p) \rightarrow H_*(X(j); \mathbb{F}_p)$  is injective, and  
 (4) for each  $j \geq N$ ,  $X(j)$  satisfies condition H(j).

We then want to construct a map  $X(k) \rightarrow X(k+1)$  which extends the sequence (2) one step to the right and satisfies the same five properties listed above. Clearly, if this can be done, then the third property implies that the map

$$H_*(X; \mathbb{F}_p) \rightarrow H_*(\text{hocolim}_k X(k); \mathbb{F}_p)$$

is injective, as desired. The first two of the four properties listed above also imply that the map  $\pi_j(\text{hocolim}_k X(k)) \cong \pi_j(X^{<N})$  is an isomorphism for  $j < N$ , and that  $\pi_j(\text{hocolim}_k X(k))$  vanishes for all  $j \geq N$ .<sup>4</sup>

Those observations are enough to show that  $X^{<N}$  agrees with any of the usual “attach cells to kill homotopy in degrees  $> N$ ” constructions applied to  $X$ , for the following reason: if  $A_N: \text{SHC} \rightarrow \text{SHC}$  is any functor and  $\eta_N: \text{id}_{\text{SHC}} \rightarrow A_N$  any natural transformation such that, for all spectra  $X$ ,  $A_N(X)$  has homotopy concentrated in degrees  $< N$  and  $\eta_N(X): X \rightarrow A_N(X)$  induces an isomorphism in homotopy in degrees  $< j$ , then applying  $A_N$  to the maps  $\eta_N(X)$  and  $X \rightarrow X^{<N}$  yields a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X^{<N} \\ \downarrow \eta_N(X) & & \cong \downarrow \eta_N(X^{<N}) \\ A_N(X) & \longrightarrow & A_N(X^{<N}) \end{array}$$

by naturality of  $\eta_N$ . The bottom horizontal map induces an isomorphism

$$\pi_j(A_N(X)) \rightarrow \pi_j(A_N(X^{<N}))$$

for  $j < N$ , and both sides vanish if  $j \geq N$ , so the bottom horizontal map is an isomorphism in SHC. So  $A_N(X) \cong X^{<N}$ . So although we will not construct  $X^{<N}$  by simply attaching cells as one does classically, our spectrum  $X^{<N}$  will indeed be isomorphic to the result of that classical construction.

We construct  $X(k+1)$ , and the map  $X(k) \rightarrow X(k+1)$ , as follows: choose a set  $I_k$  of elements of  $\pi_k(X(k))$  with the property that the image of  $I_k$  in  $\pi_k(X(k))/p$  is a *minimal* set of generators for the group  $\pi_k(X(k))/p$ . Then we apply Lemma 2.5, letting the  $I, X, n$ , and  $n$  in the statement of Lemma 2.5 be  $I_k, X(k), k$ , and  $k+1$ , respectively. We check that the hypotheses of Lemma 2.5 are satisfied:

- for each  $i \in I_k$ , the map  $f_i: S^k \rightarrow X(k)$  is zero in mod  $p$  homology by the  $j = k$  case of the fourth inductive hypothesis (see Remark 2.3), above,
- the map (1) is an isomorphism, by the minimality hypothesis on  $I_k$  as a set of generators for  $\pi_k(X(k)) \otimes_{\mathbb{Z}} \mathbb{F}_p$ ,
- and  $X(k)$  satisfies condition H(k) by the fourth inductive hypothesis.

Now we define  $X(k+1)$  to be the cofiber of the map

$$f: \left( \coprod_{i \in I_k} S^k \right)_{H\mathbb{F}_p} \longrightarrow (X(k))_{H\mathbb{F}_p} \xrightarrow{\cong} X(k)$$

<sup>4</sup>This is because each  $X(i)$  is  $H\mathbb{F}_p$ -nilpotently complete and has the property that  $\pi_{i-1}X(i)/p$  is trivial, and so  $\pi_{i-1}X(i)$  is trivial; this is a typical argument about Ext- $p$ -completeness.

given by  $H\mathbb{F}_p$ -nilpotent completion and the universal property of the coproduct applied to the maps  $\{f_i : i \in I_k\}$ , just as in the statement of Lemma 2.5. By construction, the first and second inductive hypotheses then hold for  $j = k + 1$ . Lemma 2.5 then gives us that the third inductive hypothesis also holds in the case  $j = k + 1$ , and that  $X(k + 1)$  satisfies condition  $H(k)$  and hence also condition  $H(k + 1)$ <sup>5</sup>, so the fourth hypothesis is also true when  $k$  is replaced by  $k + 1$ . This completes the induction.

The resulting map  $X \rightarrow \text{hocolim}_k X(k)$  is an isomorphism in homotopy in degrees  $< N$ , and injective in mod  $p$  homology, as desired. We still need to show that  $\pi_j(\text{hocolim}_k X(k))$  vanishes for all  $j \geq N$ . By construction,  $\pi_j(\text{hocolim}_k X(k))/p \cong \pi_j(X(j + 1))/p$  is trivial for all  $j \geq N$ . Since  $X(j + 1)$  is also  $H\mathbb{F}_p$ -nilpotently complete, the group  $\pi_j(X(j + 1))$  is Ext- $p$ -complete by Theorems 2.5 and 6.6 of [9]. The vanishing mod  $p$  of an Ext- $p$ -complete abelian group implies the vanishing of that abelian group, as a special case of Theorem A.6(d) of [18]; so  $\pi_j(X(j + 1)) \cong \pi_j(\text{hocolim}_k X(k)) \cong 0$  for all  $j \geq N$ , as desired.  $\square$

### 3. MORAVA K-THEORY OF HOMOTOPY LIMITS

Recall that, for each prime number  $p$  and positive integer  $n$ , we have the homotopy fiber sequence

$$\Sigma^{2(p^n-1)}k(n) \rightarrow k(n) \rightarrow H\mathbb{F}_p,$$

and the composite map

$$(5) \quad H\mathbb{F}_p \rightarrow \Sigma^{2p^n-1}k(n) \rightarrow \Sigma^{2p^n-1}H\mathbb{F}_p$$

is the cohomology operation  $Q_n$ , which satisfies  $Q_n^2 = 0$ . This implies some useful relationships between Morava  $K$ -theories and Margolis homology of  $E(Q_n)$ -modules, which we summarize in an appendix to this paper, appendix A, containing various results which are basically well-known but which are not well-documented in the literature. That appendix does not logically depend on anything earlier in the paper. The reader who is not already familiar with Margolis homology and its topological applications can consult appendix A for a crash course.

**Definition 3.1.** *We say that an  $E(Q_n)$ -module is  $Q_n$ -acyclic if the Margolis  $Q_n$ -homology  $H(M; Q_n)$  vanishes. We say that a morphism of  $E(Q_n)$ -modules is a  $Q_n$ -equivalence if it induces an isomorphism in  $Q_n$ -Margolis homology.*

Recall the following useful result of Adams, Theorem III.15.2 of [1]:

**Theorem 3.2.** *Suppose that  $R$  is a subring of  $\mathbb{Q}$ ,  $E$  is a bounded-below spectrum such that  $H_r(E)$  is a finitely generated  $R$ -module for all  $r$ , and  $\{Y_i\}_{i \in I}$  is a set of spectra such that  $\pi_r(Y_i)$  is an  $R$ -module for all  $r$ . Suppose that there exists a uniform lower bound for  $\pi_*(Y_i)$ , i.e., there exists an integer  $N$  such that  $\pi_n(Y_i) \cong 0$  for all  $n < N$ . Then the canonical map of spectra*

$$E \wedge \prod_{i \in I} X_i \rightarrow \prod_{i \in I} (E \wedge X_i)$$

*is a weak equivalence.*

As an easy corollary of Theorem 3.2:

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<sup>5</sup>It is easy to verify, from Definition 2.2, that any spectrum satisfying condition  $H(k)$  also satisfies condition  $H(k + 1)$ .

**Corollary 3.3.** *Let*

$$\cdots \rightarrow Y_2 \rightarrow Y_1 \rightarrow Y_0$$

*be a sequence of morphisms of spectra. Suppose that there exists a uniform lower bound on  $\pi_*(Y_i)$ . Then the canonical map of spectra*

$$H\mathbb{F}_p \wedge \operatorname{holim}_i X_i \rightarrow \operatorname{holim}_i (H\mathbb{F}_p \wedge X_i)$$

*is a weak equivalence.*

*Consequently, for each nonnegative integer  $n$ , we have a short exact sequence*

$$0 \rightarrow R^1 \lim_i H_{n+1}(X_i; \mathbb{F}_p) \rightarrow H_n(\operatorname{holim}_i X_i; \mathbb{F}_p) \rightarrow \lim_i H_n(X_i; \mathbb{F}_p) \rightarrow 0.$$

*The dual action of  $Q_n$  on mod  $p$  homology is compatible with the comparison map (3.2), so that the direct sum of the sequences (3.3),*

$$0 \rightarrow \Sigma^{-1} R^1 \lim_i H_*(X_i; \mathbb{F}_p) \rightarrow H_*(\operatorname{holim}_i X_i; \mathbb{F}_p) \rightarrow \lim_i H_*(X_i; \mathbb{F}_p) \rightarrow 0,$$

*is a short exact sequence of graded  $E(Q_n)$ -modules.*

*Proof.* We have the commutative diagram in the stable homotopy category

$$\begin{array}{ccccc} E \wedge \operatorname{holim}_i X_i & \longrightarrow & E \wedge \prod_i X_i & \xrightarrow{\operatorname{id}-T} & E \wedge \prod_i X_i \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{holim}_i (E \wedge X_i) & \longrightarrow & \prod_i (E \wedge X_i) & \xrightarrow{\operatorname{id}-T} & \prod_i (E \wedge X_i) \end{array}$$

whose rows are homotopy fiber sequences. The middle and right-hand vertical maps in (3) are weak equivalence, by the special case  $R = \mathbb{Z}$ ,  $E = H\mathbb{F}_p$  of Theorem 3.2. The map of long exact sequences induced in  $\pi_*$  by (3), together with the Five Lemma, now imply that the left-hand vertical map in (3) is also a weak equivalence.

The sequence (3.3) is simply the Milnor exact sequence for  $\pi_* \operatorname{holim}_i (H\mathbb{F}_p \wedge X_i)$ .  $\square$

**Lemma 3.4.** *Let  $p$  be a prime and let  $X$  be a bounded spectrum (i.e.,  $X$  is bounded above and bounded below) such that  $H_*(X; \mathbb{F}_p)$  is finite type. Then the  $E_\infty$ -page of the Adams spectral sequence*

$$(6) \quad E_2^{s,t} \cong \operatorname{Ext}_{\operatorname{gr} \operatorname{Comod}(A_*)}^{s,t}(\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p)) \Rightarrow \pi_{t-s} \left( (k(n) \wedge X)_{H\mathbb{F}_p} \right) \\ d_r: E_r^{s,t} \rightarrow E_r^{s+r, t+r-1}$$

*is  $v_n$ -power-torsion. That is, every bihomogeneous element  $x$  in that  $E_\infty$ -page satisfies  $v_m x = 0$  for some  $m$ .*

*Proof.* Since  $X$  is bounded, it is in particular bounded below, so  $k(n) \wedge X$  is also bounded below. Since  $k(n)$  is a ring spectrum with  $\pi_0(k(n)) \cong \mathbb{F}_p$ ,  $\pi_*(k(n) \wedge X)$  is a  $\mathbb{F}_p$ -vector space, hence is automatically  $p$ -adically complete. So  $k(n) \wedge X$  is  $H\mathbb{F}_p$ -nilpotently complete, so the  $E_\infty$ -page of spectral sequence (6) is the associated graded of a filtration on  $k(n)_*(X)$ . Since  $X$  is bounded, it is  $K(n)$ -acyclic, so  $v_n^{-1}(k(n)_*(X))$  must vanish. The associated graded of a filtered  $\mathbb{F}_p[v_n]$ -module  $M$  can have fewer  $v_n$ -torsion-free elements than  $M$ , but it cannot have more, so inverting  $v_n$  on the  $E_\infty$ -page of spectral sequence (6) must yield zero.  $\square$

Finally, we come to the main technical tool obtained in this paper.

**Theorem 3.5.** *Let  $n, N, M$  be integers such that  $n > 0$  and  $N \geq M$ , and let*

$$\cdots \rightarrow Y_2 \rightarrow Y_1 \rightarrow Y_0$$

*be a sequence of morphisms of bounded-below  $H\mathbb{F}_p$ -nilpotently complete spectra. Make the following assumptions:*

- *the graded  $\mathbb{F}_p$ -vector spaces  $H_*(Y_i; \mathbb{F}_p)$  and  $H_*(Y_i^{<N}; \mathbb{F}_p)$  are finite type for all  $i$ ,*
- *there is an isomorphism  $\lim_i H(H_*(Y_i; \mathbb{F}_p); Q_n) \cong 0$ , and*
- *each spectrum  $Y_i$  satisfies condition  $H(M)$ .*

*Then  $\text{holim}_i Y_i$  is  $K(n)$ -acyclic.*

*Proof.* We prove this in several steps:

Step 1: We consider the homotopy fiber sequence

$$K(n) \wedge \text{holim}_i Y_i^{\geq N} \rightarrow K(n) \wedge \text{holim}_i Y_i \rightarrow K(n) \wedge \text{holim}_i Y_i^{<N}$$

in which the right-hand term is contractible since bounded-above spectra are  $K(n)$ -acyclic for  $n > 0$ . So  $\text{holim}_i Y_i$  is  $K(n)$ -acyclic if and only if  $\text{holim}_i Y_i^{\geq N}$  is  $K(n)$ -acyclic. So we only need to show that  $\text{holim}_i Y_i^{\geq N}$  is  $K(n)$ -acyclic. We therefore need to show that  $k(n)_*(\text{holim}_i Y_i^{\geq N})$  is  $v_n$ -power-torsion. Here we note that since the spectra  $Y_i^{\geq N}$  are uniformly bounded below there is a weak equivalence

$$k(n) \wedge (\text{holim}_i Y_i^{\geq N}) \simeq \text{holim}_i (k(n) \wedge Y_i^{\geq N})$$

by Corollary 3.3. It therefore suffices to show that  $\pi_*(\text{holim}_i k(n)_* Y_i^{\geq N})$  is  $v_n$ -power-torsion. Since  $H_*(Y_i^{\geq N}; \mathbb{F}_p)$  is finite type and  $k(n) \wedge Y_i^{\geq N}$  is bounded below and  $H\mathbb{F}_p$ -nilpotent, we can apply [22, Prop. 2.2] to produce a strongly convergent inverse-limit Adams spectral sequence of the form

$$(7) \quad \lim_i \text{Ext}_{\text{gr Mod}(E(Q_n))}^{*,*} (H^*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p) \Rightarrow \pi_*(\text{holim}_i k(n) \wedge Y_i^{\geq N}).$$

We therefore need to prove:

- (1) that the  $E_\infty$ -page of (7) is  $v_n$ -power-torsion, and
- (2) there cannot exist an infinite sequence of nontrivial extensions in the  $E_\infty$ -page of (7) which could yield a non- $v_n$ -power-torsion element in  $\pi_*(\text{holim}_i k(n) \wedge Y_i^{\geq N})$ .

Step 2: The second task, to show that we cannot have an infinite sequence of nontrivial  $v_n$ -extensions, is easy as we will now show explicitly. The  $E_2$ -page in question is isomorphic to

$$(8) \quad \begin{aligned} & \lim_i \text{Ext}_{\text{gr Comod}(A_*)}^{*,*} (\mathbb{F}_p, H_*(k(n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y_i^{\geq N}; \mathbb{F}_p)) \\ & \cong \lim_i \text{Ext}_{\text{gr Comod}(A_*)}^{*,*} (\mathbb{F}_p, (A_* \square_{E(Q_n)_*} \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y_i^{\geq N}; \mathbb{F}_p)) \end{aligned}$$

$$(9) \quad \cong \lim_i \text{Ext}_{\text{gr Comod}(E(Q_n)_*)}^{*,*} (\mathbb{F}_p, H_*(Y_i^{\geq N}; \mathbb{F}_p))$$

$$(10) \quad \cong \lim_i \text{Ext}_{\text{gr Mod}(E(Q_n))}^{*,*} (H^*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p)$$

with isomorphism (8) by the well-known description of  $H_*(k(n); \mathbb{F}_p)$  given by [7], isomorphism (9) by a standard change-of-rings isomorphism as in Corollary A1.3.13 of [28], and isomorphism (10) by the assumption that  $H_*(Y_i^{\geq N}; \mathbb{F}_p)$  is finite type and  $Y_i^{\geq N}$  is bounded below.

Since every graded  $E(Q_n)$ -module is a coproduct of cyclic  $E(Q_n)$ -modules (a special case of the Cohen-Kaplansky theorem, from [12]), we know

$$\mathrm{Ext}_{\mathrm{grMod}(E(Q_n))}^{*,*} (H^*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p)$$

is a direct product of copies of

$$(11) \quad \mathrm{Ext}_{\mathrm{grMod}(E(Q_n))}^{*,*} (\Sigma^m \mathbb{F}_p, \mathbb{F}_p) \quad \text{and}$$

$$(12) \quad \mathrm{Ext}_{\mathrm{grMod}(E(Q_n))}^{*,*} (\Sigma^{m'} E(Q_n), \mathbb{F}_p)$$

for various integers  $m$  and  $m'$ . The Ext-module (11) is isomorphic to  $\mathbb{F}_p[v_n]$ , while the Ext-module (12) is isomorphic to  $\mathbb{F}_p[v_n]/v_n$ , concentrated on the cohomological degree 0 line.

Since  $Y_i^{\geq N}$  is  $N$ -connective,  $H^*(k(n) \wedge Y_i^{\geq N}; \mathbb{F}_p)$  vanishes below grading degree  $N$ . So, while we may have a copy of  $\mathbb{F}_p[v_n]$  generated in bidegree  $(0, N)$  in the  $E_2$ -page of the Adams spectral sequence for  $Y_i^{\geq N}$ , the  $E_2$ -page of that spectral sequence vanishes above the line formed by such a potential  $v_n$ -tower, i.e.,  $E_2^{s,t} \cong 0$  if  $t < (2p^n - 1)s + N$ . Of course the  $E_\infty$ -page must then vanish above the same line. So, given an element  $x \in E_\infty^{s,t}$ , although we may have that  $v_n x = 0$  in  $E_\infty$  but  $v_n x \neq 0$  in  $k(n)_*(Y_i^{\geq N})$  due to a filtration jump in the  $E_\infty$ -page, a fixed choice of element  $x$  cannot support an infinite sequence of such filtration jumps, since after finitely many such filtration jumps the resulting element in  $E_\infty$  would necessarily be above the vanishing line, and hence would be zero. Write  $E_r^{*,*}(Y_i^{\geq N})$  for the  $E_r$ -page of the Adams spectral sequence for  $Y_i^{\geq N}$  and simply  $E_r^{*,*}$  for the  $E_r$ -page of (7). Then there are isomorphisms

$$\lim_i E_r^{*,*}(Y_i^{\geq N}) \cong E_r^{*,*},$$

by the construction of the inverse limit Adams spectral sequence (see remark above Proposition 2.2 in [22]). Since there is a uniform vanishing line for each Adams spectral sequence for  $Y_i^{\geq N}$ , we have the same vanishing line in (15) and the same argument implies that there cannot be any infinite  $v_n$ -towers after resolving multiplicative extensions.

Step 3: Therefore, it suffices to prove that the  $E_\infty$ -page of (7) is  $v_n$ -power torsion. By Theorem 2.6, the sequence

$$(13) \quad 0 \rightarrow H_*(Y_i; \mathbb{F}_p) \rightarrow H_*(Y_i^{< N}; \mathbb{F}_p) \rightarrow H_*(\Sigma Y_i^{\geq N}; \mathbb{F}_p) \rightarrow 0$$

is exact for each  $i$ . Since  $H_*(Y_i; \mathbb{F}_p)$  and  $H_*(Y_i^{< N}; \mathbb{F}_p)$ , and consequently  $H_*(Y_i^{\geq N}; \mathbb{F}_p)$ , are finite type, and since  $H(H_*(X; \mathbb{F}_p); Q_n)$  is a subquotient of  $H_*(X)$  for any spectrum  $X$ , we know that we know that  $H(H_*(Y_i; \mathbb{F}_p); Q_n)$ ,  $H(H_*(Y_i^{< N}; \mathbb{F}_p); Q_n)$ , and  $H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n)$  are each finite type. Since  $R^1 \lim$  vanishes on sequences of finite-dimensional vector spaces, we now have

$$\begin{aligned} R^1 \lim H(H_*(Y_i; \mathbb{F}_p); Q_n) &\cong R^1 \lim H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n) \\ &\cong R^1 \lim_i H(H_*(Y_i^{< N}; \mathbb{F}_p); Q_n) \cong 0. \end{aligned}$$

We apply Margolis homology to the sequence (13) to get a long exact sequence for each  $i$ , and consequently an inverse sequence of long exact sequences. In general, applying  $\lim$  to an inverse sequence of long exact sequences yields a chain complex with no guarantee of exactness anywhere at all. But in our case, we have just shown

that  $R^1 \lim_i H(-; Q_n)$  vanishes on each term in (13), so applying  $\lim_i$  does produce a long exact sequence. In this long exact sequence, we know that every third term is some suspension of

$$\lim_i H(H_*(Y_i; \mathbb{F}_p); Q_n).$$

Since this vanishes by assumption, we get an isomorphism

$$(14) \quad \lim_i H(H_*(\Sigma^{-1}Y_i^{<N}; \mathbb{F}_p); Q_n) \rightarrow \lim_i H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n).$$

induced by the right-hand map in (13).

We note that  $Y_i^{<N}$  is a bounded spectrum and  $k(n) \wedge Y_i^{<N}$  is therefore bounded below and  $H\mathbb{F}_p$ -nilpotent complete so we may use functoriality of [22, Prop. 2.2] to produce a map of inverse limit Adams spectral sequences from

$$(15) \quad \lim_i \text{Ext}_{\text{gr Mod}(E(Q_n))}^{*,*} (H^*(\Sigma^{-1}Y_i^{<N}; \mathbb{F}_p), \mathbb{F}_p) \Rightarrow \pi_*(\text{holim}_i k(n) \wedge (\Sigma^{-1}Y_i^{<N}))$$

to the spectral sequence (7).

We then observe that this map of spectral sequences induces the composite isomorphism

$$(16) \quad \begin{aligned} & \lim_i \text{Ext}_{\text{gr Comod}(E(Q_n)^*)}^{s+1,t} (H^*(\Sigma^{-1}Y_i^{<N}; \mathbb{F}_p), \mathbb{F}_p) \\ & \cong \lim_i \left( \Sigma^{-s(2(p^n-1))} H(H_*(\Sigma^{-1}Y_i^{<N}; \mathbb{F}_p); Q_n) \right)^{-(t+2p^n-1)} \end{aligned}$$

$$(17) \quad \cong \lim_i \left( \Sigma^{-s(2(p^n-1))} H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n) \right)^{-(t+2p^n-1)}$$

$$(18) \quad \cong \lim_i \text{Ext}_{\text{gr Mod}(E(Q_n))}^{s+1,t} (H_*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p)$$

with isomorphisms (16) and (18) due to Proposition A.4 from the appendix on Margolis homology, and with isomorphism (17) given by (14). To see that this isomorphism is the same as the the map of  $E_2$  pages induced by the map

$$\Sigma^{-1}Y_i^{<N} \rightarrow Y_i^{\geq N},$$

as we have claimed, we must simply observe that the isomorphism (16) is a natural isomorphism. Also, to understand the grading shift of Margolis homology in isomorphism (16) and isomorphism (18) we point the reader to our grading convention in Conventions A.2, though this grading shift plays no significant role in the argument. Note that we are not claiming that these two spectral sequences have isomorphic  $E_2$  pages, but rather that their  $E_2$  pages are isomorphic above the  $s = 0$ -line.

By Lemma 3.4 the  $E_\infty$ -page of the  $H\mathbb{F}_p$ -Adams spectral sequence for  $k(n) \wedge Y_i^{<N}$  is  $v_n$ -power-torsion for each  $i$ . To draw the same conclusion for the spectral sequence (7), we need more information. In fact, spectral sequence (15) has a vanishing line of slope  $1/|v_n|$  that crosses the  $x$ -axis at  $N$ , as we argued earlier. Consequently, the  $E_2$ -page of the inverse-limit Adams spectral sequence (7) vanishes above the  $s = 0$ -line in the region to the left of this vanishing line, by the sequence of isomorphisms above. Since the  $E_\infty$ -page of (7) is a limit of  $v_n$ -power-torsion  $E_\infty$ -pages, the only possible problem would be if a sequence of  $v_n$ -power-torsion elements became  $v_n$ -power-torsion free in the limit, but the vanishing region makes this impossible. Therefore, the  $E_\infty$ -page of (7) and consequently the  $E_\infty$ -page of (15) is  $v_n$ -power-torsion.  $\square$

4. A HIGHER CHROMATIC HEIGHT ANALOGUE OF MITCHELL’S THEOREM

In [26], Mitchell proved that  $K(m)_*(K(\mathbb{Z})) \cong 0$  for  $m \geq 2$  and consequently  $K(m)_*(K(R)) \cong 0$  for any  $H\mathbb{Z}$ -algebra  $R$ . Throughout this section we will write  $BP\langle n \rangle$  for the truncated Brown-Peterson spectrum with coefficients  $\pi_*(BP\langle n \rangle) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ . We also use the convention that  $BP\langle -1 \rangle$  is  $H\mathbb{F}_p$ . Since  $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$ , we may consider the following higher chromatic height analogue of Mitchell’s result.

**Question 4.1.** Suppose  $n$  is some integer,  $n \in [-1, \infty)$ . If  $R$  is a  $BP\langle n \rangle$ -algebra spectrum, then does  $K(m)_*(K(BP\langle n \rangle))$  vanish for all  $m \geq n + 2$ ?

This gives an upper bound on the chromatic complexity of  $K(BP\langle n \rangle)$  so that if there is a “red-shift” in algebraic K-theory of a  $BP\langle n \rangle$ -algebra spectra, then this shift is a shift of at most one. The main goal of this section is to answer this question for all  $(n, p)$  such that  $BP\langle n \rangle$  can be modeled by an  $E_\infty$  ring spectrum, which is known to be  $n = -1, 0, 1$  at all primes and  $n = 2$  at  $p = 2, 3$ . The cases  $n = -1$  and  $n = 0$  are clear since  $H\mathbb{F}_p$  and  $H\mathbb{Z}_p$  are  $E_\infty$  ring spectra. There is an  $E_\infty$  ring spectrum model for  $BP\langle 1 \rangle$  by McClure-Staffeldt [25] and there is an  $E_\infty$  ring spectrum model for  $BP\langle 2 \rangle$  at the prime  $p = 2$  by Lawson-Naumann [21] and at  $p = 3$  by Hill-Lawson [16].

**Conventions 4.2.** When  $n = -1, 0, 1$  at all primes and  $n = 2$  at the primes  $p = 2$  and  $p = 3$ , we write  $BP\langle n \rangle$  for an  $E_\infty$  ring spectrum model for the truncated Brown-Peterson spectrum, which exist by the discussion above.

**Remark 4.3.** One can ask whether our positive answer to Question 4.1, Theorem 4.10, in this section can be extended to those values of  $(n, p)$  for which  $BP\langle n \rangle$  is *not* known to admit an  $E_\infty$  ring structure—or even those values of  $(n, p)$  for which  $BP\langle n \rangle$  is known *not* to admit an  $E_\infty$  ring structure, when  $n \geq 4$  by Lawson [20] at  $p = 2$  and Senger [29] for  $p > 2$ .

We suspect that the answer is “yes.” Our arguments in this section are far from using the full strength of an  $E_\infty$  multiplication on  $BP\langle n \rangle$ : if the results of Bruner-Rognes [11] can be extended to the setting of  $E_3$  ring spectra leading to the same computations of  $H_*^c(TC^-(BP\langle n \rangle); \mathbb{F}_p)$  and  $H_*^c(TP(BP\langle n \rangle); \mathbb{F}_p)$  as appear there, then our main theorem in this section, Theorem 4.10, will apply for any  $(n, p)$  such that  $BP\langle n \rangle$  has a model as an  $E_3$  ring spectrum. The spectrum  $BP$  was shown to admit an  $E_4$  ring spectrum model by Basterra-Mandell [8]. We have heard other mathematicians claim that the spectra  $BP\langle n \rangle$  also admit  $E_3$  multiplications, but as far as we have been able to tell, this is a folklore result that does not appear in the literature. There is currently work in progress of Hahn and Wilson that would show that  $BP\langle n \rangle$  is indeed  $E_3$  as a  $BP$ -algebra.

If  $BP\langle n \rangle$  admits an  $E_3$  multiplication, and if the calculations of [11] can be made to work using only the  $E_3$  Dyer-Lashof-Kudo-Araki operations rather than the classical ( $E_\infty$ ) Dyer-Lashof-Kudo-Araki operations, then our Theorem 4.10 would give a positive answer to Question 4.1 for all primes  $p$  and heights  $n$ .

We first briefly recall the setup for topological periodic cyclic homology and topological negative cyclic homology. Let  $R$  be an  $E_1$  ring spectrum and write  $\mathbb{T}$  for the circle group. Recall that the topological Hochschild homology of  $R$ , denoted  $THH(R)$  has a canonical  $\mathbb{T}$ -action. We define *topological negative cyclic homology*

as the homotopy fixed-point spectrum

$$TC^-(R) := THH(R)^{h\mathbb{T}}$$

and *topological periodic cyclic homology* as the Tate spectrum

$$TP(R) := THH(R)^{t\mathbb{T}}.$$

There is a homological  $\mathbb{T}$ -homotopy fixed point spectral sequence

$$(19) \quad E_2^{*,*}(R) = H^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*^c(TC^-(R); \mathbb{F}_p)$$

with abutment

$$H_*^c(TC^-(R); \mathbb{F}_p) := \lim_{k \rightarrow -\infty} H_*(TC^-(R)[k]; \mathbb{F}_p)$$

where

$$TC^-(R)[k] := F(E\mathbb{T}^{(k)}, THH(R))^{\mathbb{T}}$$

and  $E\mathbb{T}^{(k)}$  is the  $k$ -skeleton of  $E\mathbb{T}$ . There is also a homological  $\mathbb{T}$ -Tate spectral sequence

$$(20) \quad \hat{E}_2^{*,*}(R) = \hat{H}^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*^c(TP(R); \mathbb{F}_p)$$

with abutment

$$H_*^c(TP(R); \mathbb{F}_p) := \lim_{k \rightarrow -\infty} H_*(TP(R)[k]; \mathbb{F}_p)$$

where

$$(21) \quad TP(R)[k] := \left( \widetilde{E\mathbb{T}} / \widetilde{E\mathbb{T}}_k \wedge F(E\mathbb{T}, THH(R)) \right)^{\mathbb{T}}$$

is the Greenlees filtration [15] where  $\widetilde{E\mathbb{T}}_k$  is the cofiber of the map  $E\mathbb{T}_+^{(k)} \rightarrow S^0$  for  $k \geq 0$  and the Spanier-Whitehead dual of  $\widetilde{E\mathbb{T}}_{-k-1}$  if  $k < 0$ .

By considering the filtrations on  $E\mathbb{T}$  and on  $\widetilde{E\mathbb{T}}$  only in a range of dimensions, there is also a spectral sequence

$$(22) \quad E_2^{*,*}(R) = H_{\geq -2k}^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*(TC^-(R)[k]; \mathbb{F}_p)$$

whose input is  $P_{k+1}(t) \otimes H_*(THH(R); \mathbb{F}_p)$  with  $|t| = -2$  and a spectral sequence

$$(23) \quad \hat{E}_2^{*,*}(R) = \hat{H}_{\geq -2k}^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*(TP(R)[-k]; \mathbb{F}_p)$$

whose input is  $P(t^{-1})\{t^k\} \otimes H_*(THH(R); \mathbb{F}_p)$  for  $k \geq 0$ .

More generally, for a homology theory  $E_*$  we write

$$E_*^c(TP(R)) := \lim_{k \rightarrow -\infty} E_*(TP(R)[k]) \text{ and}$$

$$E_*^c(TC^-(R)) := \lim_{k \rightarrow -\infty} E_*(TC^-(R)[k]).$$

We now recall that by Angeltveit-Rognes [4, Thm. 5.12], there is an isomorphism

$$H_*(THH(BP\langle n \rangle); \mathbb{F}_p) \cong H_*(BP\langle n \rangle; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots, \sigma\bar{\xi}_{n+1}) \otimes P(\sigma\bar{\tau}_{n+1})$$

of  $A_*$ -comodules with  $A_*$ -coaction

$$\nu_n : H_*(THH(BP\langle n \rangle); \mathbb{F}_p) \rightarrow A_* \otimes H_*(THH(BP\langle n \rangle); \mathbb{F}_p)$$

given by the restriction of the coproduct of  $A_*$  to  $H_*(BP\langle n \rangle; \mathbb{F}_p) \subset A_*$  on elements in  $H_*(BP\langle n \rangle; \mathbb{F}_p)$ , and by the formula

$$\nu_n(\sigma x) = (1 \otimes \sigma)(\nu_n(x))$$



in [4, Eq. 511] for elements of the form  $\sigma\bar{\xi}_i$  for  $1 \leq i \leq n$  and  $\sigma\bar{\tau}_{n+1}$ , and then for the remaining elements by the formula

$$\nu_n(xy) = \nu_n(x)\nu_n(y).$$

Bruner-Rognes [11] then compute  $H_*^c(TC^-(BP\langle n \rangle); \mathbb{F}_p)$  and  $H_*^c(TP(BP\langle n \rangle); \mathbb{F}_p)$ , as we recall below. We will focus on odd primes for simplicity, but essentially the same results will hold at the prime 2. In [11], they compute

$$\begin{aligned} E_\infty^{*,*}(BP\langle n \rangle) &\cong P(t) \otimes P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \otimes \\ &E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n) \oplus T \end{aligned}$$

where  $T$  consists of classes  $x$  in filtration  $s = 0$  with  $tx = 0$  and

$$\bar{\tau}'_{k+1} = \bar{\tau}_{k+1} - \bar{\tau}_k(\sigma\bar{\tau}_k)^{p-1}$$

for  $k \geq m$ . One can easily deduce the computation

$$\begin{aligned} \hat{E}_\infty^{*,*}(BP\langle n \rangle) &\cong P(t, t^{-1}) \otimes P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \otimes \\ &E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n) \end{aligned}$$

for topological periodic cyclic homology from their computation. There are no possible additive extensions because the abutment is an  $\mathbb{F}_p$ -vector space.

It will also be useful to record the computations for each spectrum in the filtration:

$$\begin{aligned} \hat{E}_\infty^{*,*}(BP\langle n \rangle)[-k] &= [P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \otimes \\ &E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n)] \oplus V(n, k) \end{aligned}$$

where

$$(24) \quad V(n, k) := [\cap \{\ker d_{s,t} | s = -2k, t \geq 0\}]$$

and we write  $d_{s,t}$  for the differential in the spectral sequence computing

$$H_*^c(TP(BP\langle n \rangle); \mathbb{F}_p)$$

rather than the truncated one. We now compute the ‘‘continuous Margolis homology,’’ with respect to the Greenlees filtration, of the homology groups of the topological periodic homology of  $BP\langle n \rangle$ ; i.e., the limit of the Margolis homology of  $H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p)$  as  $k$  goes to negative infinity.

**Proposition 4.4.** *There are isomorphisms*

$$\lim_{k \rightarrow -\infty} H(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m) \cong 0$$

for all  $m \geq n + 2$ .

*Proof.* First, we note that  $Q_m$  acts trivially on any  $P(t^{-1})$  module contained in  $P(t^{\pm 1})$  for all  $m \geq 0$  because  $t$  is in an even degree. We also claim that the tensor product

$$\begin{aligned} &P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \otimes E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes \\ &P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n) \otimes V(n, k) \end{aligned}$$

in the abutment of the truncated homological  $\mathbb{T}$ -Tate spectral sequence is still a tensor product of  $Q_m$ -modules, in other words, we will argue that there are no possible hidden  $E(Q_n)$ -module extensions.

To see this we note that there is no room for hidden  $Q_m$ -module extension in the truncated homological  $\mathbb{T}$ -Tate spectral sequence for the sphere spectrum

$$P(t)\{t^{k-1}\} \Rightarrow H_*(TP(S)[-k])$$

which maps to the truncated homological  $\mathbb{T}$ -Tate spectral sequence for  $BP\langle n \rangle$ . The truncated  $\mathbb{T}$ -Tate spectral sequence for the sphere spectrum collapses and has abutment  $P(t^{-1})\{t^{k-1}\}$  with trivial  $Q_m$ -action. The map to the abutment  $H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p)$  is an injection and it is a map of  $Q_n$ -modules so the elements  $t^k$  must have trivial  $Q_m$ -action. Here we use the fact that the abutment is a graded  $\mathbb{F}_p$ -module so there are no hidden additive extensions involving  $t^j$ . Therefore, all elements of the form  $xt^j$  in the abutment must have  $Q_m$ -action  $Q_m(xt^j) = Q_m(x)t^j + xQ_m(t^j) = Q_m(x)t^j$ . So the  $Q_m$ -action preserves the filtration in the truncated homological  $\mathbb{T}$ -Tate spectral sequence. Since hidden comodule extensions would shift filtration, we know that the abutment is a tensor product of  $E(Q_m)$ -modules and it is isomorphic to the  $E_{\infty}$ -page as  $E(Q_m)$ -modules.

Now that we have proven the claim, we can apply the Künneth isomorphism for Margolis homology [23, Prop. 18.1.2(a)] and the fact that Margolis homology commutes with coproducts of  $E(Q_m)$ -modules to produce the isomorphism

$$\begin{aligned} H(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m) &\cong (P(t^{-1})\{t^{k-1}\} \otimes H(M_1, Q_m) \otimes H(M_2, Q_m)) \\ &\oplus H(V(n, k), Q_m)\{t^k\} \end{aligned}$$

where

$$\begin{aligned} M_1 &:= P(\bar{\xi}_{k+1} | k \geq n+1) \otimes E(\bar{\tau}'_{k+1} | k \geq n+1), \\ M_2 &:= P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n), \end{aligned}$$

and  $V(n, k)$  is defined as in (24), above. We also recall that  $Q_m(\bar{\tau}_{k+1}) = \bar{\xi}_{k-m}^{p^{m+1}}$  for  $k \geq m+1$ . This holds by the coaction

$$\nu(\bar{\tau}_{k+1}) = \bar{\tau}_m \otimes \bar{\xi}_{k-m}^{p^{m+1}} + \bar{\nu}(\bar{\tau}_{k+1})$$

where the  $\bar{\nu}(\bar{\tau}_{k+1})$  consists of terms such that  $\bar{\tau}_m \otimes x$  does not appear for any nontrivial  $x$ . It is easy to observe that the same will be true of  $\tau'_{k+1}$  for  $k \geq n+1$ . Therefore, we can compute  $Q_{k+1}$ -Margolis homology of  $M_1 \otimes M_2$  as the tensor product of the chain complex

$$(c_{k+1}) := (\mathbb{F}_p\{1\} \leftarrow \mathbb{F}_p\{\tau'_{k+1}\})$$

with the chain complex whose Margolis homology is  $H(\bar{M}_1 \otimes M_2, Q_{k+1})$  where  $M_1 \cong \bar{M}_1 \otimes E(\bar{\tau}_{k+1})$ . Since the chain complex  $(c_{k+1})$  is acyclic, we observe that the Margolis homology of  $P(t^{-1})\{t^{k-1}\} \otimes M_1 \otimes M_2$  is trivial for  $k \geq n+1$ . We also observe that the map

$$H_*(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m) \rightarrow H_*(H_*(TP(BP\langle n \rangle)[k+1]; \mathbb{F}_p), Q_m)$$

maps the summand  $H_*(V(n, k), Q_m)$  to zero and maps zero to  $H_*(V(n, k+1), Q_m)$  by examination of the map of truncated Tate spectral sequences and consequently the map is the zero map. Therefore,

$$\{H(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m)\}$$

is pro-equivalent to the constant pro-object on the zero object and in particular

$$\lim_{k \rightarrow -\infty} H(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m) = 0.$$

□

**Lemma 4.5.** *Let  $R$  be a connective  $p$ -local  $E_1$  ring spectrum such that  $\pi_0(R) \cong \mathbb{Z}_{(p)}$  and such that  $\pi_*(R)$  is finite type as a graded  $\mathbb{Z}_{(p)}$ -module. Then the homotopy groups of the enveloping algebra*

$$\pi_*(R \wedge R^{\text{op}})$$

are also a finite type graded  $\mathbb{Z}_{(p)}$ -module.

*Proof.* We first show that  $\pi_*(R \wedge_{S_{(p)}} R^{\text{op}})$  is finite type as a  $\mathbb{Z}_{(p)}$ -module. We apply the Künneth spectral sequence

$$\text{Tor}_{*,*}^{\pi_*(S_{(p)})}(\pi_*(R), \pi_*(R^{\text{op}})) \Rightarrow \pi_*(R \wedge_{\hat{S}_{(p)}} R^{\text{op}}).$$

Since  $\pi_*(R)$  is a finite type  $\mathbb{Z}_{(p)}$ -module and  $\pi_*(S)$  is a finite type  $\mathbb{Z}_{(p)}$ -module and the unit map  $S_{(p)} \rightarrow R$  is an isomorphism on  $\pi_0$ , we may see by an easy resolution argument that each bidegree in the spectral sequence is a finite type  $\mathbb{Z}_{(p)}$ -module. Since this is a first quadrant spectral sequence graded with the Serre convention, we know that only finitely many bidegrees contribute to each homotopy degree in the abutment.

Note that  $R \wedge_{S_{(p)}} R^{\text{op}}$  is defined as the equalizer

$$(25) \quad R \wedge_{S_{(p)}} \wedge R^{\text{op}} \xrightarrow[\cong]{\substack{(\psi_R)_{(p)} \wedge 1 \\ 1 \wedge (\psi_L)_{(p)}}} R \wedge R^{\text{op}}$$

where  $(\psi_R)_{(p)}$  and  $(\psi_L)_{(p)}$  are the usual right and left action of  $S_{(p)}$  on the  $p$ -local spectrum  $R$  respectively. We may then simply observe that Bousfield localization at the Moore spectrum  $S\mathbb{Z}_{(p)}$  is smashing so the equalizer (25) is equivalent to the equalizer

$$S_{(p)} \wedge (R \wedge S \wedge R^{\text{op}}) \xrightarrow[\cong]{\substack{1 \wedge \psi_R \wedge 1 \\ 1 \wedge 1 \wedge \psi_L}} S_{(p)} \wedge (R \wedge R^{\text{op}}),$$

which in turn is equivalent to  $(R \wedge R^{\text{op}})_{(p)}$  where we write  $(-)_{(p)}$  for Bousfield localization at  $S\mathbb{Z}_{(p)}$ . Since  $R \wedge R^{\text{op}}$  is already  $p$ -local, there is an equivalence to  $R \wedge R^{\text{op}} \simeq (R \wedge R^{\text{op}})_{(p)}$  and consequently,

$$\pi_*(R \wedge R^{\text{op}}) \cong \pi_*(R \wedge_{S_{(p)}} R^{\text{op}})$$

as graded  $\mathbb{Z}_{(p)}$ -modules. □

We now show that the filtration  $\{TP(BP\langle n \rangle)[i]\}$  satisfies the remaining hypotheses of Theorem 3.5.

**Lemma 4.6.** *The graded  $\mathbb{Z}_{(p)}$ -modules  $\pi_*(TP(BP\langle n \rangle)[i])$  are finite type for all  $i$  and all pairs  $(n, p)$  such that  $BP\langle n \rangle$  is an  $E_\infty$  ring spectrum.*

*Proof.* Since the truncated Tate spectral sequence

$$\hat{H}_{\geq -2k}^*(\mathbb{T}; \pi_*(THH(BP\langle n \rangle))) \Rightarrow \pi_*(TP(BP\langle n \rangle)[-k])$$

has a vanishing line of vertical slope with the Serre convention, which implies only finitely many bidegrees contribute to each homotopy degree in the abutment, it suffices to show that  $\pi_*(THH(BP\langle n \rangle))$  is a finite type graded  $\mathbb{Z}_{(p)}$ -module for all  $n$ . To show this, we apply [2, Lem. 5.2.5], which states that if  $R$  is a connective  $E_\infty$

ring spectrum whose homotopy groups  $\pi_*(R \otimes R)$  are finite type  $\pi_0(R)$ -modules then, in particular,  $THH_*(R)$  is finite type. Note that

$$\pi_*(BP\langle n \rangle) \cong \hat{\mathbb{Z}}_{(p)}[v_1, \dots, v_n]$$

is a connective commutative graded  $\pi_0(BP\langle n \rangle) \cong \mathbb{Z}_{(p)}$ -module and by assumption  $BP\langle n \rangle$  is  $E_\infty$ , so to apply [2, Lem. 5.2.5] it suffices to show that

$$\pi_*(BP\langle n \rangle \wedge BP\langle n \rangle)$$

is finite type, but this follows by Lemma 4.5.  $\square$

**Remark 4.7.** The only place where the  $E_\infty$  ring spectrum structure is used in the result above is in the hypothesis of [2, Lem. 5.2.5]. However, [2, Lem. 5.2.5] can be easily generalized to  $E_1$  ring spectra when we are only applying it to topological Hochschild homology, and not to the ‘‘higher THH’’ constructions obtained by tensoring with simplicial sets other than the standard simplicial circle, which were under consideration in the paper [2] and which motivated the assumption of an  $E_\infty$  ring structure in that paper.

**Lemma 4.8.** *The spectra  $TP(BP\langle n \rangle)[-k]$  satisfy condition  $H(M)$  for all  $k \geq 2$  and a fixed  $M$  depending on  $n$ , but not on  $k$ .*

*Proof.* It suffices to check that the sub- $A_*$ -comodule

$$\mathrm{Hom}_{\mathcal{A}}(\mathbb{F}_p, H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p)) \subset H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p)$$

of  $A_*$ -comodule primitives is bounded above. We first choose a cofinal sequence that always contains elements in homotopy that are not  $t$ -divisible to make the argument easier. Recall that

$$\begin{aligned} \hat{E}_\infty^{*,*}(BP\langle n \rangle)[-k] &= (P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \otimes \\ &E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n)) \oplus \\ &(H_*(THH(BP\langle n \rangle); \mathbb{F}_p) / \mathrm{im}(d_{-2k+2, *}^2)) \end{aligned}$$

where we assume  $k > 1$ .

We first need to rule out certain hidden  $A_*$ -comodule extensions. We claim that after resolving hidden comodule extensions there cannot be additional comodule primitives. Suppose

$$x \in H_*(TP(BP\langle n \rangle)[-k]; \mathbb{F}_p)$$

is not a comodule primitive, and has coaction

$$\nu_n(x) = \sum x_{(1)} \otimes x_{(2)},$$

adapting Sweedler’s notation used for coproducts in coalgebras to this context. Then after resolving hidden  $A_*$ -comodule extensions, we will have

$$\nu_n(x) = \sum x_{(1)} \otimes x_{(2)} + \sum y_{(1)} \otimes y_{(2)},$$

and the terms in the sum  $\sum y_{(1)} \otimes y_{(2)}$  must be in higher filtration in the spectral sequence’s  $E_\infty$ -page, and therefore cannot cancel with terms in the sum  $\sum x_{(1)} \otimes x_{(2)}$ . Thus, we cannot have an element that is not a comodule primitive in the  $E_\infty$ -page become a comodule primitive after resolving hidden comodule extensions. It therefore suffices to consider the comodule primitives in  $\hat{E}_\infty^{*,*}(BP\langle n \rangle)[k]$ , which by the argument above, are the same as the comodule primitives in the abutment.

Now, we know that the only  $A_*$ -comodule primitive in

$$P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \subset A_*$$

is the element 1 because it is a sub-Hopf algebra of the dual Steenrod algebra and its coaction is the restriction of the coalgebra structure on  $A_*$ . Also,

$$E(\bar{\tau}'_{k+1}|k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n)$$

is clearly bounded above by the same  $m$  independent of  $k$ . Also,  $t^j$  for  $j \neq 0$  is not a comodule primitive except for  $t^j$  such that  $2j > k$ , which is in degree  $-k+2$  or lower. To see this, note that the coaction

$$\psi: \hat{H}^*(S^1, \mathbb{F}_p) \rightarrow \hat{H}^*(\mathbb{T}, \mathbb{F}_p) \otimes A_*$$

of the dual Steenrod algebra on  $\hat{H}^*(\mathbb{T}, \mathbb{F}_p)$  is

$$\psi(t) = 1 \otimes t + \bar{\xi}_1 \otimes t^2 + \bar{\xi}_2 \otimes t^4 + \dots$$

when  $j > 0$  and can be computed by the formula

$$\psi(t) \cdot \psi(t^{-1}) = 1 \otimes 1$$

when  $j < 0$ . The coaction

$$\psi_k: \hat{H}_{\geq -2k}^*(\mathbb{T}; \mathbb{F}_p) \rightarrow \hat{H}_{\geq -2k}^*(\mathbb{T}; \mathbb{F}_p) \otimes A_*$$

can then be computed as

$$\psi_k(t) = \psi(t) \pmod{t^{k+1}}.$$

So the only way a power  $t^k$  could be a comodule primitive is when  $k = 0$  or  $2j > k+1$ . Consequently, the coaction of  $t^j x$  for

$$x \in (P(t^{-1})\{t^{k-1}\}) \otimes P(\bar{\xi}_k^p | 1 \leq k \leq n+1) \otimes P(\bar{\xi}_{k+1} | k \geq n+1) \otimes E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n)$$

will always have a summand

$$\bar{\xi}_1^j \otimes t^{2j} x$$

when  $k$  is sufficiently large so  $t^j x$  cannot be a comodule primitive unless  $2j > k$ . Therefore the only comodule primitives are elements in

$$E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n),$$

possibly plus some correcting terms, or elements in

$$E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n) \{t^j\}$$

such that  $2j > k$  plus some correcting terms. We claim that these are both bounded above by the maximal degree of

$$E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n).$$

This follows since  $t^j$  is in a negative degree and if

$$y \in E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n)$$

is not a comodule primitive then it will be of the form

$$y \mapsto 1 \otimes y + \sum y_1^{(1)} \otimes y_2^{(2)},$$

but then  $y_1^{(1)} \otimes t^j y_2^{(2)}$  will be nonzero, since  $t^j$  is not a zero divisor. Therefore,  $t^j y$  will not be a comodule primitive unless  $y$  is also a comodule primitive, which proves the claim.

We still need to show that the comodule primitives in

$$H_*(THH(BP\langle n \rangle); \mathbb{F}_p) / \text{im}(d_{-2k+2, *}^2),$$

are bounded above, which is not immediately obvious because the comodule primitives in

$$H_*(THH(BP\langle n \rangle))$$

are not bounded above. However, the elements  $(\sigma(\bar{\tau}_n))^k$  for  $k \geq 0$  are in  $\text{im}(d^2)$  and all remaining potential comodule primitives in  $H_*(THH(BP\langle n \rangle); \mathbb{F}_p)$  are bounded above by the same bound as the highest degree element in the  $A_*$ -comodule

$$E(\bar{\tau}'_{k+1} | k \geq n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leq k \leq n).$$

Therefore, it remains to check that sums of homogeneous elements in the abutment coming from different filtrations in the truncated  $\mathbb{T}$ -tate spectral sequence are not comodule primitives. We divide into two cases. First, consider  $t^j x$  where  $x$  is a not comodule primitive. If  $x$  is not a comodule primitive, then the coaction on  $x$  is

$$1 \otimes x + \sum_{i \geq 1} x_i^{(1)} \otimes x_i^{(2)}$$

and there cannot be a term in the coaction on  $t^{j+k} y$  that cancels out the term  $x_1^{(1)} \otimes x_1^{(2)} t^j$  in the coaction on  $t^j x$ . Second, consider the case  $t^j x$  where  $x$  is a comodule primitive. The only comodule primitives are products of  $\sigma \bar{\xi}_i$  or possibly  $\tau'_{k+1}$  plus some additional classes. In each case, there will be a term of the form  $\bar{\xi}_1^{2j} \otimes t^j x$  when  $j > 0$  or  $\bar{\xi}_1^{2j} \otimes 1$  if  $j > 0$  that cannot be canceled by adding on terms in higher filtration.  $\square$

**Remark 4.9.** Note that the comodule primitives are not necessarily bounded above in  $H_*(THH(BP\langle n \rangle); \mathbb{F}_p)$ , but they are bounded above in  $H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p)$  nonetheless.

**Theorem 4.10.** *When  $BP\langle n \rangle$  is an  $E_\infty$  ring spectrum, there are isomorphisms*

$$\begin{aligned} K(m)_*(TP(BP\langle n \rangle)) &\cong 0 \text{ and} \\ K(m)_*(TC^-(BP\langle n \rangle)) &\cong 0 \end{aligned}$$

for  $m \geq n+2$  for all  $n \geq 0$ .

*Proof.* The statement

$$K(m)_*(TP(BP\langle n \rangle)) \cong 0$$

follows by Theorem 3.5 together with Proposition 4.4, Lemma 4.6, and Lemma 4.8. The fact that  $K(m)_*(TC^-(BP\langle n \rangle)) \cong 0$  for  $m \geq n+2$  then follows by the fiber sequence

$$\Sigma THH(BP\langle n \rangle)_{h\mathbb{T}} \rightarrow THH(BP\langle n \rangle)^{h\mathbb{T}} \rightarrow THH(BP\langle n \rangle)^{t\mathbb{T}}$$

and the claim that

$$K(m)_*(\Sigma THH(BP\langle n \rangle)_{h\mathbb{T}}) \cong 0$$

for  $m \geq n+2$ , which we now prove. Since  $K(m)_*$  has a Künneth isomorphism, there is a Bökstedt spectral sequence

$$HH_*^{K(m)*}(K(m)_*(BP\langle n \rangle)) \Rightarrow K(m)_* THH(BP\langle n \rangle).$$

When  $m \geq n+2$  we know that  $K(m)_*(BP\langle n \rangle) \cong 0$ , so clearly the spectral sequence converges and is trivial so that

$$K(m)_*(THH(BP\langle n \rangle)) \cong 0$$

for  $m \geq 2$ . We then simply use the fact that smashing with  $K(m)$  commutes with homotopy colimits to show that

$$K(m)_*(\Sigma THH(BP\langle n \rangle)_{h\mathbb{T}}) \cong 0$$

for  $m \geq n + 2$ .  $\square$

**Corollary 4.11.** *When  $BP\langle n \rangle$  is an  $E_\infty$  ring spectrum, there are isomorphisms*

$$K(m)_*(K(BP\langle n \rangle)) \cong 0$$

for  $m \geq n + 2 \geq 1$ .

*Proof.* The result for topological cyclic homology follows by Theorem 4.10 together with the long exact sequence in Morava K-theory associated to fiber sequence

$$TC(BP\langle n \rangle)_p \rightarrow TC^-(BP\langle n \rangle)_p \rightarrow TP(BP\langle n \rangle)_p$$

of [27, Cor. 1.5]. For algebraic K-theory, the result follows by [13, Thm. 7.3.1.8 ], which produces a fiber sequence

$$K(BP\langle n \rangle)_p \rightarrow TC(BP\langle n \rangle)_p \rightarrow \Sigma^{-1}HZ_p$$

and the associated long exact sequence in Morava K-theory, since

$$K(m)_*(\Sigma^{-1}HZ_p) = 0 \text{ for } m \geq 1.$$

Note that our result does not give a new proof in the case  $n = -1$  since [13, Thm. 7.3.1.8 ] depends on the computation of algebraic K-theory of  $\mathbb{F}_p$ .  $\square$

## APPENDIX A. REVIEW OF MARGOLIS HOMOLOGY.

This appendix, which does not logically rely on anything earlier in the paper, consists of material that is well-known to users of Margolis homology. We think it is useful to provide these results, which in a few cases we do not know written references for (but are nevertheless not difficult, and certainly not new).

**A.1. Basics of Margolis homology.** Given a graded ring  $R$ , we write  $\text{gr Mod}(R)$  for the category of graded  $R$ -modules and grading-preserving  $R$ -module homomorphisms.

**Definition A.1.** *Let  $k$  be a field and let  $E(Q)$  be the exterior  $k$ -algebra on a single homogeneous generator  $Q$  in an odd grading degree  $|Q|$ . By  $Q$ -Margolis homology we mean the functor  $H(-; Q): \text{gr Mod}(E(Q)) \rightarrow \text{gr Mod}(E(Q))$  given on a graded  $E(Q)$ -module  $M$  by the quotient*

$$H(M; Q) = \left( \ker(M \xrightarrow{Q} \Sigma^{-|Q|}M) \right) / \left( \text{im}(\Sigma^{|Q|}M \xrightarrow{Q} M) \right).$$

Of course we shouldn't use the word "homology" in the phrase "Margolis homology" unless  $H(-; Q_n)$  is a functor which turns short exact sequences of  $E(Q_n)$ -modules to long exact sequences, but happily it does have this property: given a short exact sequence of graded  $E(Q)$ -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$





*Proof.* Let  $\text{pr}: E(Q) \rightarrow \Sigma^{|Q|}k$  denote the morphism of  $k$ -vector spaces given by  $\text{pr}(a + bQ) = b$  for all  $a, b \in k$ , and let  $g(M)$  denote the morphism of graded  $k$ -vector spaces given by

$$\begin{aligned} g: \underline{\text{hom}}_{E(Q)}(M, E(Q)) &\rightarrow \underline{\text{hom}}_k(M, \Sigma^{|Q|}k) \\ (g(f))(a + bQ) &= \text{pr}(f(a + bQ)) \\ &= a\beta + b\alpha, \quad \text{where } f(1) = \alpha + \beta Q. \end{aligned}$$

We claim that  $g(M)$  is actually a morphism of  $E(Q)$ -modules, where the  $E(Q)$ -action on  $\underline{\text{hom}}_k(M, \Sigma^{|Q|}k)$  is by precomposition, i.e.,  $(Qf)(m) = f(Qm)$  for  $f \in \underline{\text{hom}}_k(M, \Sigma^{|Q|}k)$ . Clearly  $g(M)$  is  $k$ -linear, so we only need to show that  $g(M)$  commutes with multiplication by  $Q$ , which is easily verified:

$$\begin{aligned} (g(M)(Qf))(a + bQ) &= a\alpha \\ &= (g(M)(f))(Q(a + bQ)), \end{aligned}$$

where  $f(1) = \alpha + \beta Q$ . The function  $g(M)$  is natural in  $M$ , i.e.,  $g$  is a natural transformation  $g: \underline{\text{hom}}_{E(Q)}(-, E(Q)) \rightarrow \underline{\text{hom}}_k(-, \Sigma^{|Q|}k)$  of the functors

$$(26) \quad \underline{\text{hom}}_{E(Q)}(-, E(Q)), \underline{\text{hom}}_k(-, \Sigma^{|Q|}k): \text{gr Mod}(E(Q)) \rightarrow \text{gr Mod}(E(Q))^{\text{op}}.$$

The domain and codomain of  $g$  are each coproduct-preserving functors (because of the  $\text{op}$  in (26)), so if we can show that the two claims

- (1) every object of  $\text{gr Mod}(E(Q))$  is a coproduct of indecomposable objects, and
- (2)  $g$  is an isomorphism when evaluated on each indecomposable object in  $\text{gr Mod}(E(Q))$ ,

are true, then we will know that  $g$  is a natural isomorphism.

The first claim is true, since every  $E(Q)$ -module decomposes as a direct sum of cyclic  $E(Q)$ -modules, e.g. by the Cohen-Kaplansky theorem, [12]. In particular, every object of  $\text{gr Mod}(E(Q))$  is isomorphic to a coproduct of suspensions of  $E(Q)$  and suspensions of  $k$ .

We have that  $(g(\Sigma^n E(Q)))(f)(a + bQ) = 0$  if and only if  $\alpha b = -a\beta$ , where  $f(1) = \alpha + \beta Q$ , so  $f \in \ker g(\Sigma^n E(Q))$  if and only if  $\alpha = 0 = \beta$ , i.e., if and only if  $f = 0$ . So  $g(\Sigma^n E(Q))$  is injective. Since  $g(\Sigma^n E(Q))$  is an injective homomorphism between finite-dimensional  $k$ -vector spaces, it is bijective. The same argument applies with  $\Sigma^n k$  in place of  $\Sigma^n E(Q)$ , so the second claim is also true. So  $g$  is a natural isomorphism.  $\square$

**Proposition A.4.** *Let  $k, E(Q), |Q|$  be as in Definition A.1. Then we have isomorphisms of graded  $E(Q)$ -modules:*

$$(27) \quad H(M^*; Q) \cong H(M; Q)^*,$$

$$(28) \quad \text{Ext}_{E(Q)}^{s,-*}(k, M) \cong \Sigma^{-s|Q|}H(M; Q) \quad \text{if } s > 0,$$

$$(29) \quad \text{Ext}_{E(Q)}^{s,-*}(M, k) \cong \Sigma^{-(s+1)|Q|}H\left(\underline{\text{hom}}_{E(Q)}(M, E(Q)); Q\right) \quad \text{if } s > 0,$$

$$(30) \quad \cong \Sigma^{-s|Q|}H(M; Q)^*.$$

*natural in the choice of graded  $E(Q)$ -module  $M$ . (The notation  $M^*$  is for the graded  $k$ -linear dual of  $M$ , i.e.,  $M^* = \underline{\text{hom}}_k(M, k)$ .)*

*Proof.* We handle each of the isomorphisms (27) through (30) in turn:

The  $Q$ -Margolis homology of  $M$  is the homology of the chain complex

$$(31) \quad \dots \xrightarrow{Q} \Sigma^{|Q|} M \xrightarrow{Q} M \xrightarrow{Q} \Sigma^{-|Q|} M \xrightarrow{Q} \dots$$

and so, since the  $k$ -linear dual of the multiplication-by- $Q$  map on a  $E(Q)$ -module is the multiplication-by- $Q$  map on the  $k$ -linear dual of that module, the cohomology of the  $k$ -linear dual of the chain complex (31) is  $H(M^*; Q)$ . So the classical universal coefficient sequence for chain complexes (e.g. as in 3.6.5 of [30]) yields the isomorphism (27).

Applying  $\underline{\mathrm{hom}}_{E(Q)}(-, M)$  to the projective graded  $E(Q)$ -module resolution of  $k$

$$(32) \quad 0 \leftarrow E(Q) \xleftarrow{Q} \Sigma^{|Q|} E(Q) \xleftarrow{Q} \Sigma^{2|Q|} E(Q) \xleftarrow{Q} \dots$$

yields the cochain complex

$$0 \rightarrow M \xrightarrow{Q} \Sigma^{-|Q|} M \xrightarrow{Q} \Sigma^{-2|Q|} M \xrightarrow{Q} \dots$$

whose homology is  $\Sigma^{-s|Q|} H(M; Q)$  in each cohomological degree  $s > 0$ . This gives us isomorphism (28). (See Convention A.2 for the sign change in grading degrees.)

We take advantage of the fact that  $E(Q)$  is self-injective, so that

$$(33) \quad 0 \rightarrow \Sigma^{-|Q|} E(Q) \xrightarrow{Q} \Sigma^{-2|Q|} E(Q) \xrightarrow{Q} \Sigma^{-3|Q|} E(Q) \xrightarrow{Q} \dots$$

is an injective graded  $E(Q)$ -module resolution of  $k$ .<sup>7</sup> Applying  $\underline{\mathrm{hom}}_{E(Q)}(M, -)$  to (33) yields the cochain complex

$$(34) \quad 0 \rightarrow \underline{\mathrm{hom}}_{E(Q)}(M, \Sigma^{-|Q|} E(Q)) \xrightarrow{Q} \underline{\mathrm{hom}}_{E(Q)}(M, \Sigma^{-2|Q|} E(Q)) \xrightarrow{Q} \dots,$$

hence isomorphism (29).

Isomorphism (30) then follows from the chain of isomorphisms

$$\begin{aligned} \Sigma^{-(s+1)|Q|} H\left(\underline{\mathrm{hom}}_{E(Q)}(M, E(Q)); Q\right) &\cong \Sigma^{-(s+1)|Q|} H\left(\underline{\mathrm{hom}}_k(M, \Sigma^{|Q|} k); Q\right) \\ &\cong \Sigma^{-(s+1)|Q|} H\left(\Sigma^{|Q|} M^*; Q\right) \\ &\cong \Sigma^{-s|Q|} H(M; Q)^*, \end{aligned}$$

due to Lemma A.3. □

### A.2. The relationships between Margolis homology and Morava $K$ -theory.

Recall that, for each prime number  $p$  and each positive integer  $n$ , we have the ring spectrum  $K(n)$ , the  $p$ -primary height  $n$  Morava  $K$ -theory spectrum, whose ring of homotopy groups is given by  $\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}]$  with  $|v_n| = 2(p^n - 1)$ . The connective cover of  $K(n)$ , written  $k(n)$  and called  $p$ -primary connective height  $n$  Morava  $K$ -theory, is also a ring spectrum, and of course has its ring of coefficients given by  $\pi_*(k(n)) \cong \mathbb{F}_p[v_n]$ . The ring spectra  $K(n)$  and  $k(n)$  each depend on a choice of prime number  $p$ , but the prime number  $p$  is traditionally suppressed from the notations for  $K(n)$  and  $k(n)$ .

<sup>7</sup>To be absolutely clear: it is not a typo that (33) has  $\Sigma^{-|Q|} E(Q)$ , not  $E(Q)$ , in cohomological degree 0. This is because the kernel of multiplication by  $Q$  on  $E(Q)$  is  $\Sigma^{|Q|} k$ , so we need to desuspend to get an injective resolution of  $k$  and not  $\Sigma^{|Q|} k$ .

**Proposition A.5.** *The mod  $p$  cohomology of the connective  $p$ -primary height  $n$  Morava  $K$ -theory is given, as a graded module over the Steenrod algebra  $A$ , by*

$$H^*(k(n); \mathbb{F}_p) \cong A \otimes_{E(Q_n)} \mathbb{F}_p,$$

where  $Q_n$  is the  $n$ th Milnor primitive in  $A$ . Meanwhile,  $H^*(K(n); \mathbb{F}_p) \cong 0$ .

*Proof.* See [7]. □

Theorem A.6, like everything else in this appendix, is basically well-known; for example, that  $X$  finite and  $H(H^*(X; \mathbb{F}_p); Q_n)$  is trivial implies that  $K(n) \wedge X$  is contractible appears as Corollary 4.9 in [17]. We do not know any written reference, however, which is stated or worked out in the level of generality of Theorem A.6.

**Theorem A.6.** *Let  $p$  be a prime number and let  $n$  be a positive integer. Let  $X$  be a spectrum with finite-type mod  $p$  homology groups, and such that  $H(H^*(X; \mathbb{F}_p); Q_n)$  vanishes.*

- *Then the homotopy colimit of the sequence*

$$(35) \quad \dots \xrightarrow{v_n} \left( k(n) \wedge \Sigma^{2p^n-2} X \right)_{H\mathbb{F}_p} \xrightarrow{v_n} \left( k(n) \wedge X \right)_{H\mathbb{F}_p} \xrightarrow{v_n} \dots$$

*is contractible.*

- *If  $X$  is furthermore assumed to be bounded below, then  $K(n) \wedge X$  is also contractible, and the mapping spectrum  $F(X, K(n))$  is also contractible.*

*Proof.* Consider the spectral sequence

$$(36) \quad \begin{aligned} E_2^{s,t} &\cong \text{Ext}_{\text{gr Comod}(A_*)}^{s,t}(\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p)), \\ &\Rightarrow \pi_{t-s} \left( (k(n) \wedge X)_{H\mathbb{F}_p} \right) \\ d_r: E_r^{s,t} &\rightarrow E_r^{s+r, t+r-1}. \end{aligned}$$

We have the change-of-rings isomorphism

$$\begin{aligned} &\text{Ext}_{\text{gr Comod}(A_*)}^{s,t}(\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p)) \\ &\cong \text{Ext}_{\text{gr Comod}(A_*)}^{s,t}(\mathbb{F}_p, E(Q_n)^* \square_{A_*} \mathbb{F}_p \otimes_{\mathbb{F}_p} H_*(X; \mathbb{F}_p)) \\ &\cong \text{Ext}_{\text{gr Comod}(E(Q_n)^*)}^{s,t}(\mathbb{F}_p, H_*(X; \mathbb{F}_p)) \\ &\cong \text{Ext}_{\text{gr Mod}(E(Q_n))}^{s,t}(H^*(X; \mathbb{F}_p), \mathbb{F}_p) \end{aligned}$$

so isomorphism (30) in Proposition A.4 gives us that, if  $H(H^*(X; \mathbb{F}_p); Q)$  vanishes, then the  $E_2$ -page of spectral sequence (36) collapses on to the  $s = 0$ -line. The same argument as above then gives us that the colimit of the sequence

$$(37) \quad \dots \xrightarrow{v_n} \pi_* \left( (k(n) \wedge \Sigma^{2p^n-2} X)_{H\mathbb{F}_p} \right) \xrightarrow{v_n} \pi_* \left( (k(n) \wedge X)_{H\mathbb{F}_p} \right) \xrightarrow{v_n} \dots$$

is zero. Compactness of the sphere spectrum gives us that (37) computes the homotopy groups of the homotopy colimit of (35).

If  $X$  is assumed to be bounded below, then the nilpotent completion map  $k(n) \wedge X \rightarrow (k(n) \wedge X)_{H\mathbb{F}_p}$  is a weak equivalence, since  $k(n)$  is a ring spectrum and so  $\pi_*(k(n) \wedge X)$  is a  $\pi_0(k(n))$ -module, i.e., an  $\mathbb{F}_p$ -module, hence  $\pi_*(k(n) \wedge X)$  is  $p$ -adically complete, hence  $k(n) \wedge X$  is  $H\mathbb{F}_p$ -nilpotently complete by Theorem 6.6 of [9]. So the homotopy colimit of (37) is weakly equivalent to the telescope

of  $v_n$  on  $k(n) \wedge X$ , i.e.,  $K(n) \wedge X$ , since smashing with  $X$  commutes with homotopy colimits. Finally, since  $K(n)$  is a field spectrum, we have  $K(n)^*(X) \cong \underline{\text{hom}}_{K(n)_*}(K(n)_*(X), K(n)_*)$  (a nice general way to prove this duality isomorphism is by using the universal coefficient theorem IV.4.5 of [14], but the result was certainly known earlier).  $\square$

The boundedness hypothesis in the statement of Theorem A.6 cannot be done away with: if  $X = K(n)$ , for example, then neither  $K(n) \wedge X$  nor  $F(K(n), K(n))$  are contractible. The sequence (35), of course, still has contractible homotopy colimit in that case since each term in that sequence is contractible.

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