DETECTING THE β-FAMILY IN ITERATED ALGEBRAIC K-THEORY OF FINITE FIELDS

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Abstract. The Lichtenbaum-Quillen conjecture (LQC) relates special values of zeta functions to algebraic K-theory groups. The Ausoni-Rognes red-shift conjectures generalize the LQC to higher chromatic heights in a precise sense. In this paper, we propose an alternate generalization of the LQC to higher chromatic heights and prove a highly nontrivial case this conjecture. In particular, if the n-th Greek letter family is detected by a commutative ring spectrum R, then we conjecture that the $n+1$-st Greek letter family will be detected by the algebraic K-theory of R. We prove this in the case $n=1$ for $R=K(\mathbb{F}_q)_p$ where $p\geq 5$ and q is a prime power generator of the units in $\mathbb{Z}/p^2\mathbb{Z}$. In particular, we prove that the commutative ring spectrum $K(K(\mathbb{F}_q)_p)$ detects the β-family. The method of proof also implies that the β-family is detected in iterated algebraic K-theory of the integers. Consequently, one may relate iterated algebraic K-theory groups of the integers to modular forms satisfying certain congruences.

Contents

1. Introduction

Following Waldhausen [36], the famous Lichtenbaum-Quillen conjecture states that the map

$$K_n(A;\mathbb{Z}/\ell\mathbb{Z})\to K_n^{\text{ét}}(A;\mathbb{Z}/\ell\mathbb{Z}) \tag{1}$$

from algebraic K-theory to étale algebraic K-theory is an isomorphism for n sufficiently large where A is a nice regular ring with ℓ invertible in A and ℓ is an odd prime [21, 28]. Since algebraic K-theory satisfies Nisnevich descent and étale algebraic K-theory satisfies étale descent, the question can be translated into the question of whether the map from motivic cohomology to étale cohomology is an isomorphism in a range. In this way, the conjecture was resolved by M. Rost and V. Voevodsky as a consequence of their proof of the Bloch-Kato conjecture[1].

Thomason showed in [33] that $K_n^{\text{ét}}(A;\mathbb{Z}/\ell\mathbb{Z}) \cong \beta^{-1}K_n(A;\mathbb{Z}/\ell\mathbb{Z})$ under the same conditions on A where β is the Bott element in $K_2(A;\mathbb{Z}/\ell\mathbb{Z})$. From the perspective of homotopy theory, we may therefore view the map (1) as the map on π_n induced by the map of spectra

$$S/\ell \wedge K(A) \to v_1^{-1}S/\ell \wedge K(A)$$

where S/ℓ is the cofiber of multiplication by ℓ. Here the map $v_1\colon\ \Sigma^{2p-2}S/\ell \to S/\ell$ is a v_1-self map which has the property that no iterate of it with itself is null homotopic. This allows us to define $v_1^{-1}S/\ell$ as the homotopy colimit of the diagram

$$S/\ell \xrightarrow{v_1} \Sigma^{-2p+2}S/\ell \xrightarrow{v_1} \Sigma^{-4p+4}S/\ell \xrightarrow{v_1} \dots$$

of spectra. The effect of inverting the Bott element is the same as the effect of inverting v_1 by work of Snaith [31] as interpreted by Waldhausen [36, Sec. 4].

The original motivation of the Lichtenbaum-Quillen conjecture was to relate algebraic K-theory groups to special values of zeta functions. For A the ring of integers in a totally real number field F and ℓ an odd prime, Wiles proved that quotients of étale cohomology groups of $A[1/\ell]$ recover special values of the Dedekind zeta function ζ_F [38]. The Lichtenbaum-Quillen conjecture then gives a correspondence between algebraic K-theory groups and special values of Dedekind zeta functions. Notably these special values correspond to the v_1-periodic part of $S/\ell_*K(A)$ because they are detected in $v_1^{-1}S/\ell_*K(A)$.

As another specific example, consider the algebraic K-theory of finite fields $\mathbb{F}_q$ where q is a prime power that topologically generates the ring $\mathbb{Z}_\ell^\times$ and ℓ is an odd prime (or equivalently q generates the units in $\mathbb{Z}/\ell^2\mathbb{Z}$). D. Quillen [27] computed $K_n(\mathbb{F}_q)$ for all n and after localizing at ℓ, there is an isomorphism

$$K_{2s-1}(\mathbb{F}_q;\mathbb{Z}_{(\ell)}) \cong \mathbb{Z}/\ell^{\nu_\ell(k)+1}\mathbb{Z}$$

where $s=(\ell-1)k$ and $\nu_\ell(k)$ is the ℓ-adic valuation of k. The order of the group $K_{2s-1}(\mathbb{F}_q;\mathbb{Z}_{(\ell)})$ corresponds exactly to the ℓ-adic valuation of the denominator of $B_s/2s$ where B_s is the s-th Bernoulli number. Recall that Bernoulli numbers are the coefficients in the Taylor series

$$\frac{x}{e^x-1} = \sum_{s\geq 0} B_s\frac{x^s}{s!}$$

and the special values of the Riemann zeta function satisfy $\zeta(-s) = (-1)^sB_s/(s+1)$ for $s\geq 0$.

This example is intimately tied to stable homotopy theory as well. J.F. Adams showed that the image of the J-homomorphism from the homotopy groups of the stable orthogonal group to the stable homotopy groups of spheres is highly nontrivial and the classical Bott periodicity in the homotopy groups of the stable orthogonal group corresponds to periodicity in the stable homotopy groups of spheres [1]. In fact, the ℓ-local image of the J-homomorphism exactly corresponds to the image of the map $\pi_*(S_{(\ell)}) \to K_{2(\ell-1)k-1}(\mathbb{F}_q;\mathbb{Z}_{(\ell)})$ when ℓ is an odd prime. The image of J therefore bridges the fields of homotopy theory and number theory. The spectrum $H\mathbb{F}_q$ detects v_0-periodicity in the sense that $\ell^k = v_0^k$ is nontrivial in the image of the Hurewicz map $\pi_*S \to \pi_*H\mathbb{F}_q$. Therefore, we have observed an instance

[1]The proof of this theorem stretches over several papers by M. Rost and V. Voevodsky. There is currently a book in progress by C. Haesemeyer and C. Weibel [18], which provides a self contained reference for the proof. For a published account see [35]. Also, see Chapter VI Theeorem 4.1 and Historical Remark 4.4 in Weibel [37] for further discussion of the state of affairs.

where algebraic K-theory of a spectrum that detects v_0-periodic elements detects v_1-periodic elements. One goal of this introduction is to formulate a precise conjecture about a higher chromatic height generalization of this phenomena. The main theorem of this paper is evidence for this conjecture at a higher chromatic height.

In chromatic stable homotopy theory, we study periodic families of elements in the homotopy groups of spheres. The first such family, due to J.F. Adams [1] and H. Toda [34], is the α-family, which consists of maps α_k defined as the composites

$$\alpha_k\colon\ \Sigma^{(2\ell-2)k}S \xrightarrow{i_0} \Sigma^{(2\ell-2)k}S/\ell \xrightarrow{v_1^k} S/\ell \xrightarrow{\delta_0} \Sigma S$$

where ℓ is an odd prime. The elements α_k are ℓ-torsion elements in the groups $\pi_{2(\ell-1)k-1}S$. These elements are in the image of the J-homomorphism at odd primes ℓ and as discussed earlier they are also detected in algebraic K-theory of finite fields of order q when q generates $(\mathbb{Z}/\ell^2\mathbb{Z})^\times$. In particular, they correspond to certain special values of the Riemann zeta function. Now, consider the cofiber of the periodic self-map $v_1\colon\ \Sigma^{2p-2}S/\ell \to S/\ell$ denoted $V(1)$. When $\ell \geq 5$, there exists a periodic self-map $v_2\colon\ \Sigma^{2\ell^2-2}V(1) \to V(1)$ and there is an associated periodic family of elements in the homotopy groups of spheres called the β-family. In particular, L. Smith [30] proved that the maps

$$\beta_k\colon\ \Sigma^{(2\ell^2-2)k}S \xrightarrow{i_0 i_1} \Sigma^{(2\ell^2-2)k}V(1) \xrightarrow{v_2^k} V(1) \xrightarrow{\delta_0\delta_1} \Sigma^{2\ell}S$$

are nontrivial. This family of elements also has a deep connection to number theory by work of Behrens [9]. In particular, Behrens showed that the (divided) β-family is related to a family of modular forms satisfying certain congruences [9, Thm. 1.3].

In the language of chromatic homotopy theory, the α-family is a periodic family of height one and the β-family is a periodic family of height two. There are a family of homology theories $K(n)_*$ called Morava K-theory which are useful for detecting periodicity of chromatic height n in the homotopy groups of spheres. The coefficients of Morava K-theory are $K(n)_* \cong \mathbb{F}_\ell[v_n^{\pm 1}]$ for $n \geq 1$ and $K(0)_*$ is rational homology. We say a ℓ-local finite cell S-module V has type n if the groups $K(n)_*V \ncong 0$ and the groups $K(n-1)_*V$ vanish. By the celebrated periodicity theorem of Hopkins-Smith [20], any ℓ-local finite spectrum V of type n admits a periodic self map

$$v_n^m\colon\ \Sigma^{(2\ell^n-2)m}V \to V.$$

We can therefore define $v_n^{-1}V$ in the same way that we defined $v_1^{-1}S/\ell$. We can also construct the n-th Greek letter family by including into the bottom cell, iterating v_n^m k-times, and then projecting onto the top cell. However, it is highly non-trivial to prove that Greek letter elements that are constructed in this way are actually nonzero.

The study of Greek letter family elements was significantly expanded by the groundbreaking work of Miller-Ravenel-Wilson [23] using the chromatic spectral sequence

$$E_1^{*,*} = \bigoplus_{i\geq 0} Ext^{*,*}_{BP_*BP}(BP_*, v_i^{-1}BP_*/(\ell^\infty, v_1^\infty, \ldots, v_{i-1}^\infty)) \Rightarrow Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \tag{2}$$

which converges to the input of the BP-Adams spectral sequence. If the class

$$v_n^k/\ell^{i_0}v_1^{i_1}\ldots v_{n-1}^{i_{n-1}} \in Ext^0_{BP_*BP}(BP_*, v_i^{-1}BP_*/(\ell, v_1, \ldots, v_{i-1}))$$

in the E_1-page of (2) survives the chromatic spectral sequence, we will write

$$\alpha^{(n)}_{k/(i_{n-1}, i_{n-2}, \ldots i_0)} \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$$

for its image in the abutment of the chromatic spectral sequence. We will refer to the collection of all such elements for a fixed n as the n-th divided (algebraic) Greek letter family and when any of the i_j for $0 \leq j \leq n-1$ are 1 we omit them from the notation. If the elements $\alpha_k^{(n)}$ survive the BP-Adams spectral sequence, then we will refer to the collection as the *n-th Greek letter family*. The advantage of this approach is that the elements in the input of the chromatic spectral sequence always exist. The question of whether or not certain Greek letter elements exist in homotopy can then be approached by determining whether certain elements in the chromatic spectral sequence and the BP-Adams spectral sequence are permanent cycles.

We will say that a (commutative) ring spectrum R *detects* the n-th Greek letter family in the homotopy groups of spheres if each element $\alpha_k^{(n)}$ is non-trivial in the image of the unit map

$$\pi_* S \longrightarrow \pi_* R.$$

We conjecture the following higher chromatic height analogue of the Lichtenbaum-Quillen conjecture, which is in the same spirit as the red-shift conjectures of Ausoni-Rognes [6]. For the following conjecture, suppose the n-th and the $n+1$-st Greek letter family are nontrivial elements in $\pi_* S$ for a given prime ℓ.

Conjecture 1.1. If R is a commutative ring spectrum that detects the n-th Greek letter family, then $K(R)$ detects the $n+1$-st Greek letter family.

We can now state the main theorem of this paper. As discussed earlier, the spectrum $K(\mathbb{F}_q)_\ell$ detects the α-family for $\ell \geq 5$ and q a prime power that generates $(\mathbb{Z}/\ell^2\mathbb{Z})^\times$. The main theorem of this paper is a proof of Conjecture 1.1 in the case $n = 1$ where $R = K(\mathbb{F}_q)_\ell$. For the following theorem, let $\ell \geq 5$ be a prime and q a prime power that generates $(\mathbb{Z}/\ell^2\mathbb{Z})^\times$. One can easily check that $\ell = 5$ and $q = 2$ is an example of such ℓ and q.

Theorem 1.2. The commutative ring spectrum $K(K(\mathbb{F}_q)_\ell)$ detects the β-family.

In particular, the method of proof also provides the following higher Lichtenbaum-Quillen-type result about iterated algebraic K-theory of the integers.

Corollary 1.3. The commutative ring spectrum $K(K(\mathbb{Z}))$ detects the β-family.

In [9], M. Behrens gives a description of the β-family in terms of modular forms satisfying certain congruences. From this point of view, our main result may be viewed as a higher chromatic height version of the Lichtenbaum-Quillen conjecture. It is therefore a step towards the larger program of understanding the arithmetic of commutative ring spectra.

The β-elements β_k that we detect only agree with the divided β-family elements $\beta_{k/i,j}$ of M. Behrens [9] when $i = j = 1$. To make the connection to arithmetic more tight, it would be desirable to detect the entire divided β-family in iterated algebraic K-theory of finite fields. It is a long term goal of the author's to show that, in fact, the entire divided β-family is detected in iterated algebraic K-theory of finite fields and consequently iterated algebraic K-theory of the integers.

1.1. **Conventions.** Let $\mathfrak{S}$ be the category of symmetric spectra in pointed simplicial sets with the positive flat stable model structure. Most of the results here can also be proven for other models of the stable homotopy category since they depend only on the homotopy category, but the proof relies on the author's joint paper with A. Salch [3] which uses this model for the stable homotopy category.

Co-modules M over a Hopf algebroid (E_*, E_*E) will always be considered with left co-action

$$\psi_M^E : M \to E_*E \otimes_{E_*} M$$

and we will simply write ψ when the module M and the Hopf algebroid (E, E_*E) is understood from the context. The main examples of interest are $E = H\mathbb{F}_p$, where E_*E is the dual Steenrod algebra $\mathcal{A}_*$, and $E = BP$. We write

$$\Delta_E : E_*E \to E_*E \otimes_{E_*} E_*E$$

for the co-product of the Hopf-algebroid E_*E or simply Δ when E is understood from the context. When $E = H\mathbb{F}_p$, this is the co-product in the dual Steenrod algebra $\mathcal{A}_* \cong P(\bar{\xi}_i \mid i \geq 1) \otimes E(\bar{\tau}_i \mid i \geq 0)$ which is defined on each algebra generator by the formulas

$$\Delta(\bar{\xi}_n) = \sum_{i+j=n} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}$$
$$\Delta(\bar{\tau}_n) = 1 \otimes \bar{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}.$$

Here, by $\bar{x}$ we mean χx where $\chi\colon \mathcal{A}_* \to \mathcal{A}$ is the antipode structure map of the Hopf-algebra $\mathcal{A}_*$. When $E = BP$, the co-product on elements of $BP_*BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots] \otimes \mathbb{Z}_{(p)}[t_1, t_2, \dots]$ is defined by the formula

$$\Delta(t_n) = \sum_{i+j=n}^{F} t_i \otimes t_j^{p^i}$$

where F is the formal group law of BP associated to the complex orientation $MU \to BP$, which equips BP with the universal p-typical formal group law. Throughout, we will write $H_*(-)$ for $H_*(-;\mathbb{F}_p)$; i.e, homology with $\mathbb{F}_p$-coefficients. Also, throughout we will work at a prime $p \geq 5$ and we will let q be a prime power that topologically generates $\mathbb{Z}_p^\times$ or equivalently generates the units in $\mathbb{Z}/p^2\mathbb{Z}$. We will write $\mathbb{Z}_p$ for p-complete integers and X_p for the p-completion of a spectrum, which agrees with the Bousfield localization $L_{S/p}X$ at the mod p Moore spectrum S/p. We will write $\doteq$ to indicate that an equality holds up to multiplication by a unit in $\mathbb{F}_p$. In the introduction, we used ℓ to denote our fixed prime because that is more closely aligned with conventions in étale cohomology, but the author is a homotopy theorist at heart and therefore can't resist using p to denote our fixed prime, which is more common in chromatic homotopy theory.

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2. Overview of the toolkit

2.1. **The THH-May spectral sequence for $K(\mathbb{F}_q)_p$.** We recall necessary results and definitions from the author's paper [2] and the author's joint paper with A. Salch [3] since they will be cited later.

Definition 2.1. A filtered commutative ring spectrum I is a cofibrant object in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ where $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ has the model structure created by the forgetful functor to $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ and $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ has the projective model structure. (See [3, Sec. 4.1] for a discussion of why these model structures exist and have the desired properties). We write I_i for I evaluated on the natural number i. The associated graded of I is defined as a commutative ring spectrum E_0I in [3] and it is defined so that, after forgetting the commutative monoid structure, it is the spectrum $\vee_{i\geq 0} I_i/I_{i+1}$ where I_i/I_{i+1} is the cofiber of the map $I_{i+1} \to I_i$. (Note that since I is cofibrant, the map $I_{i+1} \to I_i$ is a cofibration and I_i is cofibrant for each i so the cofiber agrees with the homotopy cofiber.)

Remark 2.2. This definition differs slightly from that in [3, Def 3.1.2]. A cofibrant object in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ is always a decreasingly filtered commutative monoid in $\mathfrak{S}$ in the sense of [3, Def 3.1.2] the converse is not always true. We will therefore work with a smaller category of filtered commutative ring spectra then in [3], but it will be sufficient for our purposes.

Example 2.3. As was proven in [3, Thm 4.2.1], an example of a filtered commutative ring spectrum associated to a connective commutative ring spectrum R is the Whitehead filtration

$$\cdots \to \tau_{\geq 2}R \to \tau_{\geq 1}R \to \tau_{\geq 0}R$$

which is equipped with structure maps $\rho_{i,j} : \tau_{\geq i}R \wedge \tau_{\geq j}R \to \tau_{\geq i+j}R$. Here $\tau_{\geq s}R$ is a spectrum with $\pi_i(\tau_{\geq s}R) \cong 0$ for $i < S$ that is equipped with a map $\tau_{\geq s}R \to R$ that induces an isomorphism on homotopy groups π_i for $i \geq s$. We write simply $\tau_{\geq \bullet}R$ for the filtered commutative ring spectrum constructed in [3, Thm 4.2.1] as a cofibrant object in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$.

Theorem 2.4 (Theorem 3.4.8 [3])**.** There is a spectral sequence associated to a filtered commutative ring spectrum I in topological Hochschild homology for any connective spectrum homology theory E_*

$$E^1_{*,*} = E_{*,*}(THH(E_0I)) \Rightarrow E_*(THH(I_0))$$

which we call the E-THH-May spectral sequence.

Remark 2.5. When $I = \tau_{\geq \bullet}R$ we simply write $H\pi_*R$ for E_0I. It is a generalized Eilenberg-Maclane spectrum so whenever π_kR is a finitely generated abelian group for all k and $E = S/p$, $H\mathbb{F}_p$, $V(1)$, or $BP \wedge V(1)$, then $E_*THH(E_0I)$ is a graded $H\mathbb{F}_p$-algebra and we can apply the following lemma to compute the input.

The following lemma is a consequence of the fact that all $H\mathbb{F}_p$-modules are equivalent to a wedge of suspensions of $H\mathbb{F}_p$ and an Adams spectral sequence argument, see [7] for an alternate proof.

Lemma 2.6. Let M be an $H\mathbb{F}_p$-algebra. Then M is equivalent to a wedge of suspensions of $H\mathbb{F}_p$, and the Hurewicz map

$$\pi_*M \longrightarrow H_*M$$

induces an isomorphism between π_*M and the subalgebra of $\mathcal{A}_*$-co-module primitives contained in H_*M.

Using the lemma above, we can compute the E_1-page of the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence. For details see [2].

Proposition 2.7. There is an isomorphism of $\mathcal{A}_*$-comodule algebras

$$\begin{aligned}&(H\mathbb{F}_p \wedge V(1))_*THH(H\pi_*K(\mathbb{F}_q)_p) \cong \\ &\mathcal{A}_* \otimes E(\epsilon_1) \otimes P(\tilde{v}_1) \otimes E(\sigma\bar{\xi}_1, \sigma\tilde{v}_1) \otimes P(\mu_2) \otimes HH_*(S/p_*(H\pi_*K(\mathbb{F}_q)_p))\end{aligned}$$

where the $\mathcal{A}_*$-co-action is the usual one, that is the coproduct in $\mathcal{A}_*$, on elements in $\mathcal{A}_*$ and the remaining co-actions are primitive.

We can compute the input of the $V(1)$-THH-May spectral sequence using Lemma 2.6.

Proposition 2.8 (Proposition 3.6 [2])**.** There is an isomorphism of graded $\mathbb{F}_p$-algebras

$$V(1)_*THH(H\pi_*K(\mathbb{F}_q)_p)) \cong E(\lambda_1, \epsilon_1, \sigma\tilde{v}_1) \otimes P(\mu_1, \tilde{v}_1) \otimes HH_*(S/p_*(H\pi_*K(\mathbb{F}_q)_p))$$

where $|\epsilon_1| = |\lambda_1| = |\sigma\tilde{v}_1| = 2p-1$, $|\alpha_1| = 2p-3$, $|\mu_1| = 2p$, $|\tilde{v}_1| = 2p-2$, and $|\sigma\alpha_1| = 2p-2$.

Our computations build on the computation of homology of topological Hochschild homology of $K(\mathbb{F}_q)_p$ due to Angeltveit-Rognes [4].

Theorem 2.9 (Theorem 7.13 and Theorem 7.15 [4])**.** There is an isomorphism of $\mathcal{A}_*$-comodule algebras

$$H_*K(\mathbb{F}_q)_p \cong P(\tilde{\xi}_1^p, \tilde{\xi}_2, \bar{\xi}_3, ...) \otimes E(\tilde{\tau}_2, \bar{\tau}_3, ...) \otimes E(b) \cong (\mathcal{A}//A(1))_* \otimes E(b)$$

where all the elements in $(\mathcal{A}//A(1))_*$ besides $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, and $\tilde{\xi}_2$, and b have the usual $\mathcal{A}_*$-co-action and the co-action on the remaining elements $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, $\tilde{\xi}_2$, and b are

$\psi(b) = 1 \otimes b$
$\psi(\tilde{\xi}_1^p) = 1 \otimes \tilde{\xi}_1^p - \tau_0 \otimes b + \bar{\xi}_1^p \otimes 1$
$\psi(\tilde{\xi}_2) = 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \tau_1 \otimes b + \bar{\xi}_2 \otimes 1$
$\psi(\tilde{\tau}_2) = 1 \otimes \tilde{\tau}_2 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p + \bar{\tau}_0 \otimes \tilde{\xi}_2 - \tau_1\tau_0 \otimes b + \bar{\tau}_2 \otimes 1.$

There is also an isomorphism

$$H_*THH(K(\mathbb{F}_q)_p) \cong H_*K(\mathbb{F}_q)_p \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P(\sigma\tilde{\tau}_2) \otimes \Gamma(\sigma b)$$

of $\mathcal{A}_*$-co-modules and $H_*K(\mathbb{F}_q)_p$-algebras. The $\mathcal{A}_*$-co-action is given by using the formula

$$\psi(\sigma x) = (1 \otimes \sigma) \circ \psi(x)$$

and the previously stated co-actions.

We now recall the computation of $V(1)$-homotopy of topological Hochschild homology of $K(\mathbb{F}_q)_p$ where $p \geq 5$ and q is a prime power that generates $(\mathbb{Z}/p^2\mathbb{Z})^\times$ in [2].[2]

Theorem 2.10 (Theorem 1.3 [2])**.** There is an isomorphism of graded $\mathbb{F}_p$-algebras

$$V(1)_*THH(K(\mathbb{F}_q)_p) \cong P(\mu_2) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, \alpha_1, \lambda_1', \lambda_2\alpha_1, \lambda_2\lambda_1', \lambda_2\lambda_1'\alpha_1\}.$$

where $\alpha_1 \cdot (\lambda_2\lambda_1') = \lambda_1' \cdot (\lambda_2\alpha_1) = \lambda_2\lambda_1'\alpha_1$.

2.2. **The generalized homological homotopy fixed point spectral sequence.** In this section, we summarize and extend results from Sections 2-4 of [13]. The main generalization is from $H\mathbb{F}_p$ to a connective homology theory E_* such that E is a ring spectrum and E_* is a graded $\mathbb{F}_p$-algebra.

Let $\mathbb{T} \subset \mathbb{C}^\times$ be the circle group, and let X be a $\mathbb{T}$-spectrum. We let $E\mathbb{T} = S(\mathbb{C}^\infty)$, the unit sphere in $\mathbb{C}^\infty$, where $\mathbb{T}$ acts on $\mathbb{C}$ by rotation and on $\mathbb{C}^\infty$ coordinate-wise. It is well known that there is a $\mathbb{T}$-equivariant filtration of $E\mathbb{T}_+$

$$\emptyset \hookrightarrow S(\mathbb{C})_+ \hookrightarrow S(\mathbb{C}^2)_+ \subset \ldots \hookrightarrow E\mathbb{T}_+. \tag{3}$$

such that the cofiber of each map $S(\mathbb{C}^n)_+ \hookrightarrow S(\mathbb{C}^{n+1})_+$ is $\mathbb{T}_+ \wedge S^{2n}$ for $n \geq 0$. We may produce a tower of cofiber sequences by applying the functor $F(-, X)^{\mathbb{T}}$ to the tower of $\mathbb{T}$-equivariant cofibrations (3) and, since we will take $F(E\mathbb{T}_+, X)^{\mathbb{T}}$ as our model for $X^{h\mathbb{T}}$, we have $X^{h\mathbb{T}} = \lim_k F(S(\mathbb{C}^n)_+, X)^{\mathbb{T}}$. Since, by adjunction,

$$F(\mathbb{T}_+ \wedge S^{2n}, X)^{\mathbb{T}} \cong F(S^{2n}, X) = \Sigma^{-2n}X,$$

we can apply a connective homology theory $E_*(-)$ to the tower of cofiber sequences above to produce the unrolled exact couple of E_*E co-modules

$$\begin{tikzcd}
\ldots \arrow[r] & E_*F(S(\mathbb{C}^{n+1})_+, X)^{\mathbb{T}} \arrow[r, "i"] & E_*F(S(\mathbb{C}^n)_+, X)^{\mathbb{T}} \arrow[r, "i"] \arrow[d, "j"] & E_*F(S(\mathbb{C}^n)_+, X)^{\mathbb{T}} \arrow[r] \arrow[d, "j"] & \ldots \\
\ldots & & E_*\Sigma^{-2n}X \arrow[ul, "k"] & 0 \arrow[ul, "k"] & \ldots
\end{tikzcd}$$

[2] Theorem 1.3 in [2] was also proven concurrently using a different method by Eva Höning in her Ph.D. thesis.

where j shifts degree by -1.

Then there exists a spectral sequence with input

$$E^2_{s,t} \cong \begin{cases} E_tX & \text{if } s = -2n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

which conditionally converges to

$$E^c_{s+t}X^{h\mathbb{T}} := \lim E_{s+t}F(S(\mathbb{C}^k), X)^{\mathbb{T}}.$$

which we will refer to as *continuous E-homology of $X^{h\mathbb{T}}$*. In order to identify the E^2-page in a particularly nice way, we will use an extra assumption on E.

Lemma 2.11. Suppose E_* is a graded $\mathbb{F}_p$-algebra, then the E^2 page of (4) can be identified as follows:

$$E^2_{*,*} = H^*_{gp}(\mathbb{T}, \mathbb{F}_p) \otimes E_*X.$$

where $H^*_{gp}(\mathbb{T}, \mathbb{F}_p) = H^*(B\mathbb{T}, \mathbb{F}_p) \cong P(t)$ where $|t| = -2$.

Proposition 2.12 (c.f. Proposition 2.1 and Proposition 4.1 in [13])**.** Let E be a ring spectrum such that E_* a connective graded $\mathbb{F}_p$-algebra. There is a natural homological spectral sequence of E_*E co-modules

$$E^2_{*,*} = P(t) \otimes E_*(X)$$

which strongly converges to $E^c_*(X^{h\mathbb{T}})$ when E_*X is finite type or the spectral sequence collapses at the $E^N_{*,*}$-page for some $N \geq 2$, and conditionally converges otherwise. If, in addition, X is a commutative ring spectrum, then this is a spectral sequence of E_*E-comodule algebras where E_*X has the Pontryagin product.

Proof. The proof is the same as that of [13] and therefore we omit it here. □

Proposition 2.13 (c.f Section 3 in [13])**.** Suppose $E_*THH(R)$ is a non-negatively graded graded $\mathbb{F}_p$-vector space. The d^2 differentials in the generalized homological homotopy fixed point spectral sequence associated to $THH(R)$ are of the form

$$d^2(x) = t\sigma x.$$

where t is the generator of $H^{-*}_{gp}(\mathbb{T}; \mathbb{F}_p) \cong P(t)$ in degree -2.

Proof. The proof is essentially the same as that of Bruner-Rognes and therefore we omit it. □

In the sequel, we will write $T_k(R)$ for $F(S(\mathbb{C}^k)_+, THH(R))^{\mathbb{T}}$. Note that there is a truncated homotopy fixed point spectral sequence with k columns converging to $E_*(T_k(R))$ and

$$\lim E_*T_k(R) = E^c_*(THH(R)^{h\mathbb{T}}).$$

Classically, negative cyclic homology $HC^-_*(A)$ of a commutative ring A is $\pi_*(B^{\text{cy}}_{\otimes}(A)^{h\mathbb{T}}$ where $\pi_*B^{\text{cy}}_{\otimes}(A)$ is the usual Hochschild homology of A. This lead Hesselholt [19] to coin the term topological negative cyclic homology for the $\mathbb{T}$-homotopy fixed points $THH(R)^{h\mathbb{T}}$ of topological Hochschild homology of a commutative ring spectrum R and use notation $TC^-(R)$ to denote this object. We will continue to follow this convention and also write $E^c_*(TC^-(R))$ for $E^c_*(THH(R)^{h\mathbb{T}})$.

3. Detecting the β-family in iterated algebraic K-theory of finite fields

3.1. **Detecting v_2 and β_1.** The mod p Moore spectrum S/p and the Smith-Toda complex $V(1)$ are defined so that they fit into exact triangles

$$S \xrightarrow{p} S \xrightarrow{i_0} S/p \xrightarrow{j_0} \Sigma S$$

and

$$\Sigma^{2p-2} S/p \xrightarrow{v_1} S/p \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1} S/p$$

in the stable homotopy category of spectra. We will abuse notation and write $i_1 \colon \pi_* S/p \to \pi_* V(1)$ and $i_0 i_1 \colon \pi_* S \to V(1)$ for the maps induced by i_0, i_1 and $i_0 \circ i_1$ respectively.

In the proof of the following proposition, we will will make use of differentials in both the generalized homological homotopy fixed point spectral sequence and the Adams spectral sequence. To differentiate between the two, we use notation d^r for differentials in the generalized homological homotopy fixed point spectral sequence and we use the notation d_r for differentials in the Adams spectral sequence. The following argument is inspired an argument of Ausoni-Rognes [5, Prop. 4.8].

Proposition 3.1. The classes v_2, $i_0 i_1 \beta_1$, and $i_1 \beta_1'$ in $V(1)_*$ map nontrivially to the classes $t\mu_2$, $t\sigma b$, and $t\sigma\tilde{\xi}_1^p$ respectively in $V(1)_* TC^-(K(\mathbb{F}_q)_p)$.

Proof. First, v_2 is represented by $\bar{\tau}_2 \otimes 1$, β_1' is represented by $\bar{\xi}_1^p \otimes 1$ and β_1 is represented by

$$b_{1,0} = \Sigma_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \bar{\xi}_1^i \otimes \bar{\xi}_1^j \otimes 1, \tag{5}$$

in the E_1-page of the Adams spectral sequence that converges to $\pi_* V(1)$ by [22] (cf. Section 9 of [23]). We consider the map of Adams spectral sequences

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H_* V(1)) \longrightarrow \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H_* V(1) \otimes T_2(K(\mathbb{F}_q)_p))$$

induced by the unit map

$$V(1) \wedge S \xrightarrow{1_{V(1)} \wedge \eta} V(1) \wedge T_2(K(\mathbb{F}_q)_p)) .$$

We see that $\bar{\tau}_2 \otimes 1$, $\bar{\xi}_1^p \otimes 1$, and $b_{1,0}$ are permanent cycles in the source, which map to classes of the same name in the target. Since the elements in the source are infinite cycles, this implies that the elements that they map to are infinite cycles as well. We then have to check that these classes are not boundaries.

We can eliminate the possibility of a d_1 differential with $\bar{\tau}_2 \otimes 1$ as a co-boundary by computing the differential in the cobar complex for $H_* V(1) \otimes H_* T_2(K(\mathbb{F}_q)_p)$ on each class of the correct degree. If the $\bar{\tau}_2$ is an element in $H_*(T_2(K(\mathbb{F}_q))$ with $\psi(\bar{\tau}_2) = \bar{\tau}_2 \otimes 1 + 1 \otimes \bar{\tau}_2$, then $d_1(\bar{\tau}_2) = \bar{\tau}_2 \otimes 1$. However, the two-column homological homotopy fixed point spectral sequence computing $H_* T_2(K(\mathbb{F}_q)_p)$ has a differential $d^2(\bar{\tau}_2) = t\mu_2$, by Proposition 2.13 and the fact that $\mu_2 = \sigma\bar{\tau}_2$. Therefore, the class $\bar{\tau}_2$ does not survive to $H_* T_2(K(\mathbb{F}_q)_p)$.

The only other classes in the the right degree in $H_* V(1) \otimes H_* T_2(K(\mathbb{F}_q)_p)$ to be the source of a d_1 hitting $\bar{\tau}_2 \otimes 1$ are $\sigma\tilde{\xi}_2$ and $\bar{\tau}_1 \sigma b$. However, σb is primitive so $d_1(\sigma b) = 0$. Also, we can compute directly $d_1(\sigma\tilde{\xi}_2) = \bar{\xi}_1 \otimes \sigma\tilde{\xi}_1^p + \bar{\tau}_1 \otimes \sigma b$ and $d_1(\bar{\tau}_1) = \bar{\tau}_1 \otimes 1 + \bar{\tau}_0 \otimes \bar{\xi}_1$. Therefore,

$$d_1(\alpha\bar{\tau}_1\sigma b + \beta\sigma\tilde{\xi}_2) = \alpha(\bar{\tau}_1 \otimes \sigma b + \bar{\tau}_0 \otimes \bar{\xi}_1 \sigma b) + \beta(\bar{\xi}_1 \otimes \sigma\tilde{\xi}_1^p + \bar{\tau}_1 \otimes \sigma b) \neq \bar{\tau}_2 \otimes 1$$

for any $\alpha, \beta \in \mathbb{F}_p$. Therefore, $\bar{\tau}_2 \otimes 1$ survives to the E_2-page. There are no possible longer differentials hitting $\bar{\tau}_2 \otimes 1$ because $\bar{\tau}_2 \otimes 1$ is in Adams filtration one; hence, it is a permanent cycle.

We eliminate the possibility that the class $\bar{\xi}_1^p \otimes 1$ is a boundary of a d_1 by the same method. As in the previous argument, the truncated homotopy fixed point spectral sequence converging to $H_*T_2(K(\mathbb{F}_q)_p)$ has a differential $d^2(\bar{\xi}_1^p) = \sigma\bar{\xi}_1^p$ by Proposition 2.13, so the class $\bar{\xi}_1^p$ does not survive to become a class in $H_*T_2(K(\mathbb{F}_q)_p)$. Therefore, the only classes that are in the right degree in $H_*V(1) \wedge T_2(K(\mathbb{F}_q)_p)$ to have $\bar{\xi}_1^p \otimes 1$ as their co-boundary are

$$\{\bar{\tau}_0\sigma\bar{\xi}_1^p, \sigma b.\}$$

However, $d_1(\sigma b) = 0$, since it is a co-module primitive, and

$$d_1(\bar{\tau}_0\sigma\tilde{\xi}_1^p) = 1 \otimes \bar{\tau}_0\sigma\bar{\xi}_1^p - \bar{\tau}_0 \otimes \sigma\tilde{\xi}_1^p - 1 \otimes \bar{\tau}_0\sigma\tilde{\xi}_1^p \neq \bar{\xi}_1^p \otimes 1$$

modulo boundaries. The class $\bar{\xi}_1^p \otimes 1$ is in Adams filtration one so it cannot be the target of a longer differential, therefore it is a permanent cycle.

For $b_{1,0}$, we need to check that it is not the boundary of a d_1 or a d_2, because it is in Adams filtration two. We first need to check that it is not a boundary of an element in $\mathcal{A}_* \otimes H_*V(1) \wedge T_2(K(\mathbb{F}_q)_p)$. We check the differential in the cobar complex on all the elements here in the right degree. These classes are

$$\left\{ \begin{array}{l} 1 \otimes \sigma b,\ \bar{\tau}_0 \otimes \bar{\tau}_0 t\tilde{\xi}_1^p,\ \bar{\xi}_1^{p-1} \otimes \bar{\tau}_0\bar{\tau}_1,\ \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1, \\ \bar{\xi}_1^{p-1}\bar{\tau}_0 \otimes \bar{\tau}_1,\ \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0,\ \bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_0 \otimes 1,\ \bar{\xi}_1^p \otimes 1 \end{array} \right\}$$

where $\tilde{\xi}_1^p$ has a coproduct coming from $H_*K(\mathbb{F}_q)$ and $\bar{\xi}_1^p$ has the co-action coming from the coproduct on $\mathcal{A}_*$. Recall that Milnor computed the co-action of $\mathcal{A}_*$ on

$$H^*(\mathbb{C}P^\infty, \mathbb{F}_p) \cong H^*(B\mathbb{T}; \mathbb{F}_p) \cong H^*(\mathbb{T}; \mathbb{F}_p),$$

and the co-action on the class t is

$$\psi(t) = \Sigma_{i \geq 0}\bar{\xi}_i \otimes t^{p^i},$$

see [24]. Therefore, in the input of the truncated homotopy fixed point spectral sequence computing $V(1)_*T_2(K(\mathbb{F}_q)_p)$, the $\mathcal{A}_*$ co-action on t is primitive.

We compute the differential in the cobar complex on each of the elements that could possibly have the class representing β_1 as a target:

$$\begin{aligned}
d_1(1 \otimes \sigma b) &= 1 \otimes 1 \otimes \sigma b \\
d_1(\bar{\tau}_0 \otimes \bar{\tau}_0 t\tilde{\xi}_1^p) &= \bar{\tau}_0 \otimes \bar{\tau}_0 \otimes t\tilde{\xi}_1^p + \bar{\tau}_0 \otimes \bar{\xi}_1^p \otimes t\bar{\tau}_0 + \bar{\tau}_0 \otimes \bar{\tau}_0 \otimes \bar{\tau}_0 tb \\
d_1(\bar{\xi}_1^{p-1} \otimes \bar{\tau}_0\bar{\tau}_1) &= 1 \otimes \bar{\xi}_1^{p-1} \otimes \bar{\tau}_0\bar{\tau}_1 - \Delta(\bar{\xi}_1^{p-1}) \otimes \bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-1} \otimes \psi(\bar{\tau}_0\bar{\tau}_1) \\
&= -\textstyle\sum_{i=1}^{p-2}\binom{p-1}{i}\bar{\xi}_1^{p-i-1} \otimes \bar{\xi}_1^i \otimes \bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-1} \otimes \bar{\tau}_0 \otimes \bar{\tau}_1 \\
&\quad +\bar{\xi}_1^{p-1} \otimes \bar{\tau}_0\bar{\tau}_1 \otimes 1 + \bar{\xi}_1^{p-1} \otimes \bar{\tau}_1 \otimes \bar{\tau}_0 + \bar{\xi}_1^{p-1} \otimes \bar{\tau}_0 \otimes \bar{\tau}_0\bar{\xi}_1
\end{aligned}$$

$$\begin{aligned}
d_1(\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1) &= 1 \otimes \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 - \Delta(\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1) \otimes \bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \psi(\bar{\tau}_0\bar{\tau}_1) \\
&= 1 \otimes \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1 - \textstyle\sum_{i=0}^{p-2}\binom{p-2}{i}\bar{\xi}_1^i \otimes \bar{\xi}_1^{p-i-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1 \\
&\quad - \textstyle\sum_{i=0}^{p-2}\binom{p-2}{i}\bar{\xi}_1^i\bar{\tau}_0 \otimes \bar{\xi}_1^{p-i-2}\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1 - \sum_{i=0}^{p-2}\binom{p-2}{i}\bar{\xi}_1^i\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\xi}_1^{p-i-2} \otimes \bar{\tau}_0\bar{\tau}_1 \\
&\quad - \textstyle\sum_{i=0}^{p-2}\binom{p-2}{i}\bar{\xi}_1^i\bar{\tau}_1 \otimes \bar{\xi}_1^{p-i-2}\bar{\tau}_0 \otimes \bar{\tau}_0\bar{\tau}_1 - \sum_{i=0}^{p-2}\binom{p-2}{i}\bar{\xi}_1^i\bar{\tau}_0 \otimes \bar{\xi}_1^{p-i-2}\bar{\xi}_1\bar{\tau}_0 \otimes \bar{\tau}_0\bar{\tau}_1 \\
&\quad +\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes 1 \otimes \bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0 \otimes \bar{\tau}_0 \\
&\quad +\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1 \otimes 1 + \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_1 \otimes \bar{\tau}_0
\end{aligned}$$

$$\begin{aligned}
d_1(\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1) &= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1-\Delta(\bar{\xi}_1^{p-1}\bar{\tau}_0)\otimes\bar{\tau}_1+\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\psi(\bar{\tau}_1)\\
&= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1-\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\otimes\bar{\xi}_1^{p-i-1}\bar{\tau}_0\otimes\bar{\tau}_1\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_0\otimes\bar{\xi}_1^{p-i-1}\otimes\bar{\tau}_1+\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes 1\otimes\bar{\tau}_1\\
&\quad +\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1\otimes 1\\
d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes\bar{\tau}_0) &= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes\bar{\tau}_0-\Delta(\bar{\xi}_1^{p-1}\bar{\tau}_1)\otimes\bar{\tau}_0+\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes 1\otimes\psi(\bar{\tau}_0)\\
&= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes\bar{\tau}_0-\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\otimes\bar{\xi}_1^{p-1-i}\bar{\tau}_1\otimes\bar{\tau}_0\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_0\otimes\bar{\xi}_1^{p-1-i}\bar{\xi}_1\otimes\bar{\tau}_0\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_1\otimes\bar{\xi}_1^{p-1-i}\bar{\tau}_1\otimes\bar{\tau}_0\\
&\quad +\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes 1\otimes\bar{\tau}_0+\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes\bar{\tau}_0\otimes 1\\
d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1\otimes 1) &= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1\otimes 1-\Delta(\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1)\otimes 1+\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1\otimes 1\otimes 1\\
&= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1\otimes 1-\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\otimes\bar{\xi}_1^{p-1-i}\bar{\tau}_1\bar{\tau}_0\otimes 1\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_0\otimes\bar{\xi}_1^{p-1-i}\bar{\xi}_1\bar{\tau}_0\otimes 1\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_1\otimes\bar{\xi}_1^{p-1-i}\bar{\tau}_1\bar{\tau}_0\otimes 1\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_0\otimes\bar{\xi}_1^{p-1-i}\bar{\tau}_1\otimes 1\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^{i}\bar{\tau}_1\bar{\tau}_0\otimes\bar{\xi}_1^{p-1-i}\bar{\tau}_1\otimes 1\\
&\quad +\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1\otimes 1\otimes 1\\
d_1(\bar{\xi}_1^{p}\otimes 1) &= 0.
\end{aligned}$$

If some linear combination of these elements has $b_{1,0}$ as a boundary, then there is a solution to the equation

$$\begin{aligned}
\Sigma_{i=1}^{p-1}\tfrac{1}{p}\tbinom{p}{i}\xi_1^{i}\otimes\xi_1^{p-i}\otimes 1 &= a_1d_1(1\otimes\sigma b)+a_2d_1(\bar{\tau}_0\otimes\bar{\tau}_0t\tilde{\xi}_1^{p})+a_3d_1(\bar{\xi}_1^{p-1}\otimes\bar{\tau}_0\bar{\tau}_1)\\
&\quad +a_4d_1(\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_0\bar{\tau}_1)+a_5d_1(\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1)\\
&\quad +a_6d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1\otimes\bar{\tau}_0)+a_7d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1\otimes 1)
\end{aligned}$$

for some elements $a_i\in\mathbb{F}_p$ for $1\le i\le 7$; however, no such solutions to this equation exist so we can conclude that $b_{1,0}$ is not a boundary of a d_1.

Since $b_{1,0}$ is in Adams filtration two, we still have to check that there is no d_2 differential hitting it in the Adams spectral sequence,

$$\mathrm{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p,H_*(V(1)\wedge T_2(K(\mathbb{F}_q)_p))\Rightarrow V(1)_*T_2(K(\mathbb{F}_q)_p).$$

Since a d_2 would have to have its source on the 0-line in degree $2p^2-2p-1$, it would have to be a class in $H_{2p^2-2p-1}V(1)\wedge T_2(K(\mathbb{F}_q)_p)$.

We compute

$$H_{2p^2-2p-1}V(1)\wedge T_2(K(\mathbb{F}_q)_p)\cong\mathbb{F}_p\{\bar{\tau}_0t\tilde{\xi}_1^{p}\},$$

since $d^2(b)=t\sigma b$ in the two column homotopy fixed point spectral sequence that computes $H_*T_2(K(\mathbb{F}_q)_p)$. Since $d_1(\bar{\tau}_0)=\bar{\tau}_0\otimes 1$, the Leibniz rule implies

$$d_1(\bar{\tau}_0(t\tilde{\xi}_1^{p}))=(\bar{\tau}_0\otimes 1)\cdot d_1(t\tilde{\xi}_1^{p})\neq 0,$$

since $d_1(t\tilde{\xi}_1^{p})=\bar{\xi}_1^{p}\otimes t+\bar{\tau}_0\otimes tb\neq 0$. So $\bar{\tau}_0(t\tilde{\xi}_1^{p})$ does not survive to the E_2-page and therefore it cannot support a differential hitting $b_{1,0}$. Therefore, the class $b_{1,0}$ is a permanent cycle.

We conclude the elements v_2, β_1' and β_1 map nontrivially from $V(1)_*S$ to

$$V(1)_*T_2(K(\mathbb{F}_q)_p)$$

via map induced by the unit map $S\to T_2(K(\mathbb{F}_q)_p)$. In $V(1)_*T_2(K(\mathbb{F}_q)_p)$, the only possible classes in the right degree to be v_2, β_1' and β_1 are $t\mu_2$, $t\sigma b$ and $t\sigma\bar{\xi}_1^{p}$, respectively.

The unit map factors through $V(1)_*TC^-(K(\mathbb{F}_q)_p)$, so these classes pull back to classes in $V(1)_*TC^-(K(\mathbb{F}_q)_p)$. □

Corollary 3.2. The classes $t\mu_2$, $t\sigma b$, and $t\sigma\tilde{\xi}_1^p$ are permanent cycles in the generalized homological homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, V(1)_*THH(K(\mathbb{F}_q)_p)) \Rightarrow V(1)_*THH(K(\mathbb{F}_q))^{h\mathbb{T}}.$$

in particular, $d_{2p-2}(t\mu_2) = 0$, $d_{2p-2}(t\sigma b) = 0$ and $d_{2p-2}(t\sigma\tilde{\xi}_1^p) = 0$.

Lemma 3.3. There is a differential $d_{2p-2}(t) = t^p\alpha_1$ in the homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, V(1)_*THH(K(\mathbb{F}_q)_p)) \Rightarrow V(1)_*THH(K(\mathbb{F}_q))^{h\mathbb{T}}.$$

Proof. First, we can show that there is a differential $d_{2p-2}(t) = t^p\alpha_1$ in the homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, V(1)_*K(\mathbb{F}_q)_p)) \Rightarrow V(1)_*K(\mathbb{F}_q)_p^{h\mathbb{T}}.$$

where $K(\mathbb{F}_q)_p$ has trivial $\mathbb{T}$-action, because α_1 is an attaching map in $B\mathbb{T}$. This has already been proven in [12, Theorem 3.5], so we omit the details. There is an $\mathbb{T}$-equivariant map of commutative ring spectra

$$THH(K(\mathbb{F}_q)_p) \to K(\mathbb{F}_q)_p$$

which induces a map of homotopy fixed point spectral sequences and since this map sends t to t and α_1 to α_1, the differential $d_{2p-2}(t) = \alpha_1 t^p$ also occurs in the homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, V(1)_*THH(K(\mathbb{F}_q)_p)) \Rightarrow V(1)_*THH(K(\mathbb{F}_q))^{h\mathbb{T}}.$$

Note that we could have also proven this directly by examining $\mathbb{T}$-equivariant attaching maps in $E\mathbb{T}$, but for the sake of brevity we give the simpler proof. □

Corollary 3.4. There are differentials $d_{2p-2}(\mu) = -t^{p-1}\alpha_1\mu$ and $d_{2p-2}(\sigma b) = -t^{p-1}\alpha_1\sigma b$.

Proof. This is immediate from Lemma 3.3 and Corollary 3.2. □

Now, the classes β_k have the property that in the BP-Adams spectral sequence for $V(1)$ they are represented by the classes

$$\binom{i}{2} v_2^{i-2}k_0 + iv_2^{i-1}b_{1,0},$$

where

$$k_0 = 2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{p+1} \otimes 1 - t_1^{2p} \otimes t_1 \otimes 1$$

which are in BP-Adams filtration two. We will therefore give a similar argument to the one in the proof of Proposition 3.1, except that we will work in the BP-Adams spectral sequence in order to use the fact that the classes representing β_k are in low BP-Adams filtration. To do this we must compute $BP \wedge V(1)_*T_k(K(\mathbb{F}_q)_p)$ up to possible d^4 differentials or longer.

3.2. The $BP\wedge V(1)$-THH-May spectral sequence.

In this section, we begin by computing the input of the $BP \wedge V(1)$-THH-May spectral sequence.

Lemma 3.5. There is an isomorphism of $(BP \wedge V(1))_*(H\pi_*(K(\mathbb{F}_q)_p))$-algebras

$$\begin{aligned}(6)\qquad &(BP \wedge V(1))_*THH(H\pi_*K(\mathbb{F}_q)_p) \cong \\ &P(t_1, t_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1) \otimes P(v_1, \mu_1) \otimes HH_*(S/p_*(H\pi_*K(\mathbb{F}_q)_p)),\end{aligned}$$

and the Hurewicz map

$$(BP\wedge V(1))_*THH(E_0(\tau_{\geq\bullet}K(\mathbb{F}_q)_p))\to(H\mathbb{F}_p\wedge BP\wedge V(1))_*THH(E_0(\tau_{\geq\bullet}K(\mathbb{F}_q)_p))$$

sends t_1 to $\bar{\xi}_1-\hat{\xi}_1$, where $\bar{\xi}_1$ is the generator in degree $2p-2$ of H_*BP and $\hat{\xi}_1$ is the generator in degree $2p-2$ of $H_*(V(1)\wedge THH(E_0(\tau_{\geq\bullet}K(\mathbb{F}_q)_p))$.

Proof. Recall that $V(1)\wedge THH(H\pi_*(K(\mathbb{F}_q)_p))$ is a $V(1)\wedge H\pi_*(K(\mathbb{F}_q)_p)$-algebra, and hence an $H\mathbb{F}_p$ algebra, since $V(1)\wedge H\pi_*(K(\mathbb{F}_q)_p)$ is itself an $H\mathbb{F}_p$-algebra. Thus, there is an equivalence

$$BP\wedge V(1)\wedge THH(H\pi_*(K(\mathbb{F}_q)_p))\simeq BP\wedge H\mathbb{F}_p\wedge_{H\mathbb{F}_p}V(1)\wedge THH(H\pi_*(K(\mathbb{F}_q)_p))$$

and by the collapse of the Künneth spectral sequence, the isomorphism (6) holds.

Since $BP\wedge V(1)\wedge THH(H\pi_*(K(\mathbb{F}_q)_p))$ is an $H\mathbb{F}_p$-module we can use Lemma 2.6, which states that $(BP\wedge V(1))_*THH(H\pi_*(K(\mathbb{F}_q)_p))$ includes as the co-module primitives inside of

$$(H\mathbb{F}_p\wedge BP\wedge V(1))_*THH(H\pi_*(K(\mathbb{F}_q)_p)).$$

We recall that by the Künneth isomorphism and Proposition 2.7 there is an isomorphism of graded rings

$$\begin{aligned}&(H\mathbb{F}_p\wedge BP\wedge V(1))_*THH(H\pi_*(K(\mathbb{F}_q)_p))\cong\\&H_*(BP)\otimes E(\bar{\tau}_0,\bar{\tau}_1,\lambda_1,\sigma v_1)\otimes(A//E(0))_*\otimes P(v_1,\mu_1)\otimes HH_*(S/p_*(H\pi_*(K(\mathbb{F}_q)_p)))\end{aligned}$$

where we use the notation $(\mathcal{A}//E(0))_*\cong P(\hat{\xi}_1,\hat{\xi}_2,\dots)\otimes E(\hat{\tau}_1,\hat{\tau}_2,\dots)$ and $H_*(BP)\cong P(\bar{\xi}_1,\bar{\xi}_2,\dots)$ to distinguish the two sets of generators. We also write $E(\bar{\tau}_0,\bar{\tau}_1)$ for the homology of $V(1)$. The co-action on $\bar{\xi}_i$, $\bar{\tau}_i$, $\hat{\tau}_i$ and $\hat{\xi}_i$ are the same as the coproduct in the dual Steenrod algebra, and hence for example $\bar{\xi}_1-\hat{\xi}_1$ is a co-module primitive, since

$$\psi(\bar{\xi}_1-\hat{\xi}_1)=1\otimes\bar{\xi}_1+\bar{\xi}_1\otimes 1-1\otimes\hat{\xi}_1-\bar{\xi}_1\otimes 1=1\otimes\bar{\xi}_1-1\otimes\hat{\xi}_1.$$

The co-action on the remaining elements in degrees less than $2p^2-2$ is

$$\begin{array}{ll}\psi(\alpha_1)=1\otimes\alpha_1 & \psi(\sigma v_1)=1\otimes\sigma v_1+\bar{\tau}_0\otimes\sigma\alpha_1\\ \psi(\sigma\alpha_1)=1\otimes\sigma\alpha_1 & \psi(\lambda_1)=1\otimes\lambda_1\\ \psi(\gamma_{p^k}(\sigma\alpha_1))=1\otimes\gamma_{p^k}(\sigma\alpha_1) & \psi(\mu_1)=1\otimes\mu_1+\bar{\tau}_0\otimes\lambda_1\\ \psi(v_1)=1\otimes v_1+\bar{\tau}_0\otimes\alpha_1. & \end{array}$$

and we may observe that there are no other $\mathcal{A}_*$ co-module primitives in degree $2p-2$ other than $\bar{\xi}_1-\hat{\xi}_1$ so t_1 must map to $\bar{\xi}_1-\hat{\xi}_1$. □

Proposition 3.6. There is an isomorphism of (BP_*,BP_*BP)-co-modules

$$(BP\wedge V(1))_*THH(K(\mathbb{F}_q)_p)\cong P(t_1^p,t_2,...)\otimes E(b)\otimes E(\sigma\bar{\xi}_1^p,\sigma\bar{\xi}_2)\otimes P(\mu_2)\otimes\Gamma(\sigma b)$$

where the co-action is given by

$$\begin{array}{ll}\psi(t_1^p)=1\otimes t_1^p+t_1^p\otimes 1 & \psi(\mu_2)=1\otimes\mu_2\\ \psi(t_n)=\Delta(t_n)\text{ for }n\geq 2 & \psi(\gamma_{p^k}(\sigma b))=1\otimes\gamma_{p^k}(\sigma b)\\ \psi(b)=1\otimes b & \psi(\sigma x)=(1\otimes\sigma)*\psi(x)\end{array}$$

Proof. We need to compute differentials in the $BP\wedge V(1)$-THH-May spectral sequence

$$E^1_{*,*}=(BP\wedge V(1))_{*,*}THH(H\pi_*(K(\mathbb{F}_q)_p)))\Rightarrow BP\wedge V(1)_*THH(K(\mathbb{F}_q)_p)$$

so we examine the map of spectral sequences

$$\begin{tikzcd}
(BP \wedge V(1))_{*,*}THH(H\pi_*(K(\mathbb{F}_q)_p))) \arrow[r, Rightarrow] \arrow[d, "h"] & BP \wedge V(1)_*THH(K(\mathbb{F}_q)_p) \arrow[d] \\
(H\mathbb{F}_p \wedge BP \wedge V(1))_{*,*}THH(H\pi_*(K(\mathbb{F}_q)_p)) \arrow[r, Rightarrow] & (H\mathbb{F}_p \wedge BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p).
\end{tikzcd}$$

induced by the Hurewicz map $BP \to H\mathbb{F}_p \wedge BP$. Recall from Lemma 3.5 that

$$\begin{aligned}
&(BP \wedge V(1))_*THH(H\pi_*(K(\mathbb{F}_q)_p)) \cong \\
&P(\xi_1, \xi_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1) \otimes P(v_1, \mu_1) \otimes HH_*(S/p_*(H\pi_*(K(\mathbb{F}_q)_p))).
\end{aligned}$$

We know that in the $H\mathbb{F}_p \wedge BP \wedge V(1)$-THH-May spectral sequence the classes $\bar{\xi}_i$ for $i \geq 1$ and $\bar{\tau}_j$ for $j = 0, 1$ survive to E^∞, since the output of the spectral sequence is known to be

$$\begin{aligned}
&(H\mathbb{F}_p \wedge BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p) \cong \\
&P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1) \otimes H_*(K(\mathbb{F}_q)_p) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2) \otimes \Gamma(\sigma b)
\end{aligned}$$

by Theorem 2.9 and the Künneth isomorphism. This forces the same d^1 differentials that occur in the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence and consequently there is an additive isomorphism

$$E^2_{*,*} = P(\xi_1, \xi_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1, \alpha_1) \otimes P(v_1, \mu_1) \otimes \Gamma(\sigma\alpha_1).$$

The map of spectral sequences is therefore again injective on E^2-pages. In the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence there are differentials

$$\begin{array}{lll}
d^{2p-3}(\hat{\xi}_1) = \alpha_1 & d^r(\bar{\xi}_i) = 0 & d^{2p-3}(\lambda_1) = \sigma\alpha_1 \\
d^{2p-2}(\hat{\tau}_1) = v_1 & d^{2p-3}(\mu_1) = \sigma v_1 & d^r(\bar{\tau}_i) = 0 \text{ for } i = 0, 1 \\
d^r(\hat{\tau}_i) = 0 & \text{for } i > 0 &
\end{array}$$

for $r \geq 2$ and no further differentials. Since the Hurewicz map h is injective and it sends t_1 to $\bar{\xi}_1 - \hat{\xi}_1$, the differential $d^{2p-3}(t_1)$ in the top spectral sequence can be computed using the formula

$$d^{2p-3}(t_1) = d^{2p-3}(h^{-1}(\bar{\xi}_1 - \hat{\xi}_1)) = h^{-1}d^{2p-3}(\bar{\xi}_1 - \hat{\xi}_1) = h^{-1}(\alpha_1) = \alpha_1.$$

Similarly, ϵ_1 maps to $\bar{\tau}_1 - \hat{\tau}_1$ implying $d^{2p-2}(\epsilon_1) = v_1$. Hence, in the $BP \wedge V(1)$-THH-May spectral sequence there are differentials

$$d^{2p-3}(t_1) = \alpha_1, \quad d^{2p-3}(\lambda_1) = \sigma\alpha_1, \quad d^{2p-2}(\epsilon_1) = v_1, \quad d^{2p-2}(\mu_1) = \sigma v_1.$$

On E^2-pages the map of spectral sequences induced by the Hurewicz map is again injective. Since $E^2 \cong E^\infty$ in the target spectral sequence, the same is true in the source. This implies that the $BP \wedge V(1)$-THH-May spectral sequence collapses at the E^2-page.

By examining the long exact sequence

$$BP_*V(1) \wedge K(\mathbb{F}_q)_p \to BP_*V(1) \wedge \ell \to BP_*V(1) \wedge \Sigma^{2p-2}\ell$$

we can determine that the co-action on t_1^p and t_i for $i \geq 2$ in $BP_*K(\mathbb{F}_q)_p$ is the same as the co-action on these elements in $BP_*V(1) \wedge \ell \cong P(t_1, t_2, \dots)$. Note that there is no hidden comultiplication on t_1^p since there are no classes in degrees $2p^2 - 2p - (2p - 2)$ or lower and the lowest degree element in BP_*BP is in degree $2p - 2$. The class b is the class in lowest degree and therefore it is primitive. This produces the co-action on b, t_1^p, t_i for $i \geq 2$ in $BP_*V(1) \wedge THH(K(\mathbb{F}_q)_p)$, by using the splitting of BP_*BP-co-modules

$$BP_*V(1) \wedge THH(K(\mathbb{F}_q)_p) \cong BP_*V(1) \wedge K(\mathbb{F}_q)_p \oplus BP_*V(1) \wedge \overline{THH}(K(\mathbb{F}_q)_p)$$

induced by the splitting $THH(K(\mathbb{F}_q)_p) \simeq K(\mathbb{F}_q)_p \vee \overline{THH}(K(\mathbb{F}_q)_p)$, which we have because $K(\mathbb{F}_q)_p$ is a commutative ring spectrum.

The co-action on μ_2 is primitive because $|\mu_2| = 2p^2$ and there are no classes in degrees $2p^2 - 2p + 2$ or $2p^2 - 4p + 4$ or lower and the classes in BP_*BP are in degrees congruent to zero mod $2p^n - 2$ for some n. Similarly, the co-action on λ_1' is primitive because there are no classes in degree $2p^2 - 2p + 1 - (2p - 2)$ or lower.

To determine the co-action on λ_2, note that there is an isomorphism

$$\begin{aligned} BP_*V(1) \wedge K(\mathbb{F}_q)_p &\cong P(\bar{\xi}_1^p, \bar{\xi}_2, \dots) \\ &\cong P(t_1^p, t_2, \dots) \end{aligned}$$

so $\bar{\xi}_2$ and t_2 are two names for the same basis element up to multiplication by a unit. Similarly, $\bar{\xi}_1^p$ and t_2 are two names for the same basis element up to multiplication by a unit. The operation σ gives $\lambda_2 = \sigma\bar{\xi}_2 \doteq \sigma t_2$ and $\lambda_1' \doteq \sigma t_1^p$ and we can therefore compute the co-action on λ_2 using the formula $\psi(\lambda_2) = (1 \otimes \sigma)\Delta(t_2)$, due to [4]. In other words, in $(BP \wedge V(1))_*THH(K\mathbb{F}_q)_p)$,

$$\psi(\lambda_2) \doteq 1 \otimes \lambda_2 + t_1 \otimes \lambda_1'.$$

This just leaves the classes $\gamma_{p^k}(\sigma b)$ for $k > 0$. Note that we already showed that in the input of $BP \wedge V(1)$-THH-May spectral sequence the classes $\gamma_{p^{k+1}}(\sigma\alpha_1) = \gamma_{p^k}(\sigma b)$ are primitive. Therefore, it suffices to check that there is not a hidden co-action in the THH-May spectral sequence. If the co-action contains terms of the form $x \otimes m$ where $|m| < |\gamma_{p^k}(\sigma\alpha_1)|$, then the May filtration of m must be greater or equal to the May filtration of $\gamma_{p^k}(\sigma b)$.

Suppose the May filtration of m is greater or equal to p^{k+1}, the May filtration of $\gamma_{p^k}(\sigma b)$. Then, since the only classes with positive May filtration are $\gamma_{p^j}(\sigma b)$, b, λ_1', and λ_2, the class m must be of the form

$$(\gamma_{p^j}(\sigma b))^\ell b^{\epsilon_1} \lambda_1'^{\epsilon_2} \lambda_2^{\epsilon_3} z,$$

for some possibly zero element z, where $0 \le \ell < p$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$. Write $\mathrm{mfilt}(x)$ for the May filtration of an element, then

$$\begin{array}{ll} \mathrm{mfilt}(\gamma_{p^j}(\sigma b)) = p^{j+1} & \mathrm{mfilt}(b) = 1 \\ \mathrm{mfilt}(\lambda_1') = p - 1 & \mathrm{mfilt}(\lambda_2) = 1. \end{array}$$

so j, ℓ, ϵ_1, ϵ_2, and ϵ_3 must satisfy

$$\ell p^{j+1} + \epsilon_1 + \epsilon_2(p-1) + \epsilon_3 \ge p^{k+1}. \tag{7}$$

We split into cases. If $k = 1$, then $j \ge k - 1$, and if $j = k - 1$, then the inequality (7) only holds if $\ell = p - 1$. In that case, ϵ_2 must be 1 and either ϵ_1 or ϵ_3 must be 1. Thus,

$$|(\gamma_{p^j}(\sigma b))^\ell b^{\epsilon_1} \lambda_1'^{\epsilon_2} \lambda_2^{\epsilon_3}| \ge (2p^2 - 2p)(p-1) + 2p^2 - 2p + 1 + 2p^2 - 2p - 1 = 2p^3 - 2p$$

But, $2p^3 - 2p > 2p^3 - 2p^2 = |\gamma_p(\sigma b)|$ contradicting the assumption that $|m| < |\gamma_p(\sigma b)|$. In the case $k > 1$, then the inequality (7) only holds if $j \ge k$, but if $j \ge k$, then

$$|(\gamma_{p^j}(\sigma b))^\ell| \ge 2p^{k+2} - 2p^{k+1} = |\gamma_{p^k}(\sigma b)|$$

so again m does not satisfy $|m| < |\gamma_{p^k}(\sigma b)|$. Thus, no such m such that $|m| < |\gamma_{p^k}(\sigma b)|$ and $\mathrm{mfilt}(m) \ge \mathrm{mfilt}(\gamma_p^k(\sigma b))$ exists. This implies that there are no hidden co-actions and $\gamma_{p^k}(\sigma b)$ remains a co-module primitive. □

Corollary 3.7. In the generalized homological homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, (BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p)) \Rightarrow (BP \wedge V(1))_*^c THH(K(\mathbb{F}_q)_p)$$

there are differentials

$$d^2(t_1^p)\doteq t\lambda_1'$$
$$d^2(t_2)\doteq t\lambda_2$$
$$d^2(b)=t\sigma b$$

and no further d^2 differentials besides those generated from these d^2 differentials using the Leibniz rule.

Proof. This follows from Proposition 2.13 and the fact that $\lambda_2\doteq\sigma t_2$ and $\lambda_1'\doteq\sigma t_1^p$ as discussed in the proof of Proposition 3.6. □

Remark 3.8. We will also need to know the co-action of BP_*BP on

$$BP_*V(1)\wedge T_{k+1}(K(\mathbb{F}_q)_p),$$

which is isomorphic to

$$P(\bar{\xi}_1^p,\bar{\xi}_2,\ldots)\otimes E(b)\otimes E(\sigma\bar{\xi}_1^p,\sigma\bar{\xi}_2)\otimes P(\mu_2)\otimes\Gamma(\sigma b)\otimes P(t)/t^k$$

modulo differentials. This just amounts to describing the co-action on the class t in the input of the generalized homological homotopy fixed point spectral sequence

$$H^*(\mathbb{T},BP_*V(1)\wedge THH(K(\mathbb{F}_q)_p)), \tag{8}$$

since the coaction on a subquotient of (8) is determined by the coaction on (8).

Since we know that $\psi_{H\mathbb{F}_p}(t)=\sum_{i\geq 0}\bar{\xi}_i\otimes t$ and $t_i\doteq\bar{\xi}_i$ in H_*BP, the functor that sends the $\mathcal{A}_*$-comodule $H^*(\mathbb{T},\mathbb{F}_p)$ to the BP_*BP-comodule $H^*(\mathbb{T},\mathbb{F}_p)\otimes BP_*/(p,v_1)$ produces the coaction

$$\psi_{BP}(t)=\sum_{i\geq 0}t_i\otimes t^{p^i}$$

$H^*(\mathbb{T},\mathbb{F}_p)$.

Note that there is a truncated generalized homotopy fixed point spectral sequence with input

$$E^2_{*,*}=P(t)/t^{k+1}\otimes BP_*V(1)\wedge THH(K(\mathbb{F}_q)_p) \tag{9}$$

and abutment $BP_*V(1)\wedge T_{k+1}(K(\mathbb{F}_q)_p)$.

Corollary 3.9. In the spectral sequence (9) computing $BP_*V(1)\wedge T_{k+1}(K(\mathbb{F}_q)_p)$, there is an isomorphism between $E^4_{*,*}$ and

$$\Big(P(\bar{\xi}_1^{p^2},\bar{\xi}_2^p,\bar{\xi}_3,\ldots)\otimes E(b\gamma_{p-1}(\sigma b))\otimes E(\sigma\bar{\xi}_1^p t_1^{p^2-p},\sigma\bar{\xi}_2 t_2^{p-1})\otimes P(\mu_2)\otimes\Gamma(\gamma_p(\sigma b))\otimes P(t)/t^{k+1}\Big)\oplus$$
$$\mathbb{F}_p\{\lambda_1'(t_1^p)^{j_1-1},\lambda_2 t_2^{j_2-1},\gamma_s(\sigma b),t^k(t_1^p)^{j_1},t^k(t_2)^{j_2},t^k b\gamma_{s-1}(\sigma b)|1\leq j_1<p,1\leq j_2<p,1\leq s<p\}\otimes$$
$$\Big(P(\bar{\xi}_1^{p^2},\bar{\xi}_2^p,\bar{\xi}_3,\ldots)\otimes E(b\gamma_{p-1}(\sigma b))\otimes E(\sigma\bar{\xi}_1^p t_1^{p^2-p},\sigma\bar{\xi}_2 t_2^{p-1})\otimes P(\mu_2)\otimes\Gamma(\gamma_p(\sigma b))\Big)$$

of BP_*BP co-modules, where the coaction on an element $x\in E^4_{*,*}$ is determined multiplicatively by the coaction of classes in $BP_*V(1)\wedge THH(K(\mathbb{F}_q)_p)$ and the coaction of t from Remark 3.8 modulo the differential d^2 determined in Corollary 3.7.

Proof. This is a direct consequence of Corollary 3.7 and the Leibniz rule. □

3.3. **Detecting the β family in homotopy fixed points of topological Hochschild homology.** Recall that $T_{k+1}(R)$ is defined to be the spectrum $F(S(\mathbb{C}^{k+1})_+,THH(R))^{\mathbb{T}}$. As noted before Proposition 1.4 in [13], $T_{k+1}(R)$ is a commutative ring spectrum whenever R is a commutative ring spectrum. In particular, $TC^-(R)$ is a commutative ring spectrum. We now recall a theorem, which is a consequence of computations of Ausoni-Rognes [5].

Theorem 3.10 (Ausoni-Rognes [5])**.** The classes v_2^k map to nonzero classes $(t\mu)^k$ under the unit map

$$V(1)_*S \to V(1)_*T_{k+1}(\ell_p).$$

Remark 3.11. Since we showed v_2 maps to $t\mu_2$ under the unit map $V(1)_*S \to V(1)_*T_2(K(\mathbb{F}_q)_p)$ and the maps

$$V(1)_*S \to T_{k+1}(K(\mathbb{F}_q)_p) \to V(1)_*T_{k+1}(\ell_p)$$

are ring maps for $k \geq 1$, the classes v_2^k also map to $(t\mu_2)^k$ under the unit map

$$V(1)_*S \to V(1)_*T_{k+1}(K(\mathbb{F}_q)_p).$$

We therefore know that $(t\mu)^k$ are permanent cycles in the BP-Adams spectral sequence and homotopy fixed point spectral sequences computing $V(1)_*T_{k+1}K(\mathbb{F}_q)_p$.

We will continue to use notation d^r for differentials in the generalized homological homotopy fixed point spectral sequence and d_r for differentials in the BP-Adams spectral sequence to differentiate the two.

Theorem 3.12. The elements β_i in π_*S are detected by a unit times the class

$$\binom{i}{2}(t\mu_2)^{i-2}t\lambda_1\lambda_2 + i(t\mu_2)^{i-1}t\sigma b$$

in $V(1)_*TC^-(K(\mathbb{F}_q)_p)$; i.e., the elements $\beta_i \in \pi_{(2p^2-2)i-2p}S$ map to the nonzero elements $\binom{i}{2}(t\mu_2)^{i-1}t\lambda_1\lambda_2 + i(t\mu_2)^i t\sigma b$ in $V(1)_*TC^-(K(\mathbb{F}_q)_p)$ up to multiplication by a unit.

Proof. Due to the length of this proof, we will break it into steps.

Step 1: We will show that v_2^i in the BP_*BP-cobar complex for $V(1)$ maps to $(t\mu)^i$ in the BP_*BP-cobar complex for $V(1)_*T_i(K(\mathbb{F}_q)_p))$. As discussed in Remark 3.11, we know that v_2^i maps to $(t\mu)^i$ under the map

$$V(1)_* \to V(1)_*T_{i+1}(K(\mathbb{F}_q)_p)$$

as a consequence of Theorem 3.10. By examining the map of THH-May spectral sequences induced by the unit map $\eta \wedge \mathrm{id}_{V(1)} : S \wedge V(1) \to BP \wedge V(1)$ and the subsequent map of generalized homological homotopy fixed point spectral sequences induced by this same map, we see that $(t\mu)^i$ maps to $(t\mu)^i$ under the map

$$V(1)_*T_{i+1}(K(\mathbb{F}_q)_p) \to (BP \wedge V(1))_*T_{i+1}(K(\mathbb{F}_q)_p).$$

We also know the map

$$\pi_*(\eta \wedge \mathrm{id}_{V(1)}) : \pi_*(S \wedge V(1)) \to \pi_*(BP \wedge V(1))$$

sends the class v_2^i to v_2^i since the edge-homomorphism in the BP-Adams spectral sequence is a ring homomorphism. We then use the commutative diagram

$$\begin{tikzcd} V(1)_* \arrow[r] \arrow[d] & V(1)_*T_{i+1}(K(\mathbb{F}_q)_p) \arrow[d] \\ BP_*V(1) \arrow[r] & (BP \wedge V(1))_*T_{i+1}(K(\mathbb{F}_q)_p) \end{tikzcd}$$

to determine that $v_2^i \in BP_*V(1)$ maps to $(t\mu)^i \in (BP \wedge V(1))_*T_{i+1}(K(\mathbb{F}_q)_p)$ and also in the map of exact couples of the respective BP-Adams spectral sequences.

Step 2: We recall that the class β_i is represented by

$$\binom{i}{2}v_2^{i-2}k_0 + iv_2^{i-1}b_{1,0} \mod (p, v_1) \tag{10}$$

in $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ where

$$k_0 = 2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{p+1} \otimes 1 - t_1^{2p} \otimes t_1 \otimes 1$$

and

$$b_{1,0} = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} t_1^{p-i} \otimes t_1^i \otimes 1 \tag{11}$$

due to Ravenel [29, Example 5.1.20]. We therefore need to check that the classes (10) map to permanent cycles in the BP_*BP-cobar complex for $V(1)_*T_{i+1}(K(\mathbb{F}_q)_p))$.

We begin with the element β_1. We observe that the element β_1 is represented by the class $b_{1,0}$ in the E_1-page of the BP-Adams spectral sequence for $V(1)$ and it maps to a class of the same name in the cobar complex for the BP_*BP-co-module $BP_*V(1) \wedge T_2(K(\mathbb{F}_q)_p)$; i.e. the E_1- page of the BP-Adams spectral sequence for $V(1) \wedge T_2(K(\mathbb{F}_q)_p)$. Let $\bar{b}_{1,0}$ be the element in the BP-Adams spectral sequence for the sphere spectrum that maps to $b_{1,0}$. Then $\bar{b}_{1,0}$ in the BP-Adams spectral sequence for the sphere spectrum maps to (5) in the Adams spectral sequence for the sphere spectrum by [23, Thm. 9.4] and both are permanent cycles. Therefore using the square of spectral sequences and Proposition 3.1, we know that $b_{1,0}$ is a permament cycle in the BP-Adams spectral sequence for $V(1) \wedge T_2(K(\mathbb{F}_q)_p)$.

Step 3: The class $b_{1,0}v_2^{k-1}$ represents β_k in the BP-Adams spectral sequence for $V(1)$ when $k \equiv 1 \mod p$. It maps to $b_{1,0}(t\mu)^{k-1}$ in the BP-Adams spectral sequence for $V(1) \wedge T_k(K(\mathbb{F}_q)_p)$ up to multiplication by a unit by the argument at the beginning of the proof and the fact that the cobar complex for $V(1) \wedge T_k(K(\mathbb{F}_q)_p)$ is multiplicative. Since the class representing β_{pm+1} is a permanent cycle in the BP-Adams spectral sequence for $V(1)$ (this follows from [26, Lemma 5.4]), the class $\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}t_1^i \otimes t_1^{p-i} \otimes (t\mu_2)^{pk}$ is an infinite cycle in the BP-Adams spectral sequence for $V(1) \wedge T_{pk+1}(K(\mathbb{F}_q)_p)$, but it could still be a co-boundary. It is on the two-line of the BP-Adams spectral sequence, so we just need to check that it is not the co-boundary of a d^1 or d^2 differential. Note that we will prove that, in fact, the element $b_{1,0}(t\mu_2)^{k-1}$ is never a boundary for any k. This only implies that $b_{1,0}v_2^{k-1}$ is a permanent cycle for all k if it is already an infinite cycle for all k in the BP-Adams spectral sequence for $V(1)$, which to the author's knowledge is unknown when $k \not\equiv 1 \mod p$. We will break this into two further sub-steps.

Sub-step 1: If the class $\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}t_1^i \otimes t_1^{p-i} \otimes (t\mu_2)^{k-1}$ is the co-boundary of a d^1, then there is a sum of classes $\sum_i a_i \otimes m_i \in BP_*BP \otimes_{BP_*} BP_*V(1) \wedge T_k(K(\mathbb{F}_q)_p)$ such that $d_1(\sum_i a_i \otimes m_i) \doteq b_{1,0}(t\mu_2)^{k-1}$.

Recall that the co-action on m is of the form $\psi(m) = 1 \otimes m + \sum_j a_j \otimes m_j$ where $|m_j| < |m|$. Observe that the only elements in $(BP \wedge V(1))_*T_k(K(\mathbb{F}_q)_p)$ whose co-action contains $(t\mu)^{k-1}$ as either m or m_j for some j are classes of the form $(t\mu)^{k-1}y$ for some $y \in (BP \wedge V(1))_*T_k(K(\mathbb{F}_q)_p)$ up to multiplication by a unit. The co-action of such a class is

$$\psi((t\mu)^{k-1}y) = (1 \otimes (t\mu)^{k-1})\psi(y),$$

and $\psi(y)$ must be of the form

$$\psi(y) = 1 \otimes y + z \otimes 1 + \sum b_i \otimes y_i$$

since $\psi((t\mu)^{k-1}y)$ must have $1 \otimes (t\mu)^{k-1}$ as a term. Since the only classes in $(BP \wedge V(1))_*T_k(K(\mathbb{F}_q)_p)$ that have $z \otimes 1$ as a term in their co-action for some element $z \neq 0$ are the classes t_1^p and t_i for $i \geq 2$ the class y must be a product of these. Since the internal degree of $a_i \cdot (\cdot t\mu)^{k-1}y$

must equal $(2p^2-2)k+2p^2-2p$, we have

$$|a_i\cdot(\cdot t\mu)^{k-1}y| = (2p^2-2)(k-1)+|y|+|a_i| = (2p^2-2)k+2p^2-2p$$

so, the degree of $|y|+|a_i|$ must be $2p^2-2p$. However, the class t_1^p is the element of lowest degree in the set $\{t_1^p, t_2, \dots\}$ and $|t_1^p| = 2p^2-2p$. Also, the only classes in degrees less than or equal to $2p^2-2p$ in BP_*BP are powers of v_1 and t_1. Therefore, the only options are $y = t_1^p$ and $a_i = 1$ or $a_i = t_1^{p-j}v_1^j$ for some $0 \le j \le p$ and $y = 1$. We know that

$$\Delta(t_1^{p-j}v_1^j) = v_1^j\Delta(t_1^{p-j}) = v_1^j * (t_1\otimes 1 + 1\otimes t_1)^{p-j} \text{ and}$$
$$\psi(t_1^p) = \Delta(t_1^p) = t_1^p\otimes 1 + 1\otimes t_1^p \mod p$$

So, we compute

$$\begin{aligned} d_1(t_1^{p-j}v_1^j\otimes(t\mu)^{k-1}) &= 1\otimes t_1^{p-j}v_1^j\otimes(v\cdot t\mu)^{k-1} - \Delta(t_1^{p-j}v_1^j)\otimes(t\mu)^{k-1}) \\ &\quad + t_1^{p-j}v_1^j\otimes\psi((t\mu)^{k-1}) \\ &= \bar{\Delta}(t_1^{p-j}v_1^j)\otimes(t\mu)^{k-1} \\ &\neq \Sigma_{i=1}^{p-1}\tfrac{1}{p}\tbinom{p}{i}t_1^i\otimes t_1^{p-i}\otimes(t\mu_2)^{k-1} \end{aligned}$$

and

$$\begin{aligned} d_1(1\otimes t_1^p(t\mu)^{k-1}) &= 1\otimes 1\otimes t_1^p(t\mu)^{k-1} - 1\otimes 1\otimes t_1^p(t\mu)^{k-1} \\ &\quad +1\otimes\psi(t_1^p(\mu)^{k-1}) \\ &= 1\otimes t_1^p\otimes(t\mu)^{k-1} + 1\otimes 1\otimes t_1^p(t\mu)^{k-1} \\ &\neq \Sigma_{i=1}^{p-1}\tfrac{1}{p}\tbinom{p}{i}t_1^i\otimes t_1^{p-i}\otimes(t\mu_2)^{k-1} \end{aligned}$$

where $\bar{\Delta}(a_i) = \Delta(a_i) - a_i\otimes 1 - 1\otimes a_i$. Thus, $m_i \doteq (t\mu)^{k-1}$ for at least one i.

Now, if $m_i \doteq (t\mu)^{k-1}$ for only one i, then the element a_i corresponding to m_i must have reduced co-product $\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}t_1^i\otimes t_1^{p-i}+z$ for some element $z\in BP_*BP\otimes_{BP_*}BP_*BP$, up to multiplication by a unit; i.e.,

$$\bar{\Delta}(a_i) \doteq \Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}t_1^i\otimes t_1^{p-i}+z.$$

The degree of a_i must be $2p^2-2p$, so $a_i \doteq t_1^jv_1^{p-j}$. However,

$$\begin{aligned} \bar{\Delta}(t_1^jv_1^{p-j}) &= v_1^{p-j}\bar{\Delta}(t_1^j) \\ &= v_1^{p-j}(t_1\otimes 1+1\otimes t_1)^j - 1\otimes t_1^jv_1^{p-j} - t_1^jv_1^{p-j}\otimes 1 \end{aligned}$$

and this does not equal $b_{1,0}+z$ for any j, and any element $z\in BP_*BP\otimes_{BP_*}BP_*BP$.

Suppose that $m_i \doteq (t\mu)^{k-1}$ for $i\in I$ where I contains more than one natural number. Then $\psi(\sum_{i\in I}a_i) \doteq b_{1,0}+z'$ for some possibly trivial element z' in $BP_*BP\otimes_{BP_*}BP_*BP$. However, we checked in the proof of Proposition 3.1 that no class of the form $\sum_{i\in I}a_i$ has co-action $b_{1,0}+z'$ and the same proof applies here.

Thus, there is no sum of classes $\sum_i a_i\otimes m_i$ such that $d_1(\sum_i a_i\otimes m_i) \doteq b_{1,0}(t\mu_2)^{k-1}$ and therefore the class $b_{1,0}(t\mu_2)^{k-1}$ survives to the E_2-page.

Sub-step 2: Now suppose there is a class in bidegree $(2p^2k-2k+2p^2-2p+1,0)$ that is the source of a d^2 differential hitting $b_{1,0}(t\mu_2)^{k-1}$. This class is therefore in $BP_{2p^2k-2k+2p^2-2p+1}V(1)\wedge T_k(K(\mathbb{F}_q)_p)$. Since this class is in an odd degree, we can classify all the classes that could possibly be in this degree as a linear combination of elements in the three families,

$$\{\lambda_1'z_1, \lambda_2z_2, t^{k-1}bz_3\}$$

where z_1 and z_2 are some nontrivial product of even dimensional classes and z_3 is some nontrivial product of even dimensional classes that does not include $t\sigma b$ or $(t\mu_2)^j$ for any $j \geq 1$ as a factor since $t\sigma b \cdot t^{k-1}bz_3 = t\mu_2 \cdot t^{k-1}bz_3 = 0$.

We can explicitly compute $d_1(\lambda_2) = \bar{\xi}_1 \otimes \lambda_1'$ modulo differentials in the generalized homological homotopy fixed point spectral sequence. Therefore, by the Leibniz rule, $d_1(\lambda_2 z_2) = (\bar{\xi}_1 \otimes \lambda_1')z_2 + \lambda_2 d_1(z_2) \neq 0$. Therefore, the classes of the form $\lambda_2 z_2$ do not survive to the E_2 page and cannot be the source of a d_2 differential hitting $b_{1,0}(t\mu_2)^{k-1}$.

We therefore just need to check elements of the form $t^k bz_3$ or $\lambda_1' z_1$ where z_3 does not contain $t\mu$ or $t\sigma b$ as a factor. Note that the Leibniz rule implies

$$d_2(t^k bz_3) = d_2(t^k b)z_3 + t^k bd_2(z_3)$$

and similarly,

$$d_2(\lambda_1' z_2) = d_2(\lambda_1')z_2 + \lambda_1' d_2(z_2)$$

so we need to check if

$$\alpha(d_2(tb)z_3 + tbd_2(z_3)) + \beta(d_2(\lambda_1')z_2 + \lambda_1' d_2(z_2)) = (t\mu)^k t\sigma b$$

for some $\alpha, \beta \in \mathbb{F}_p$. However, note that the internal degree of $d_2(\lambda_1)$ is $2p^2 - 2p + 2$ and there are no classes in that degree in $BP_*V(1) \wedge T_{k+1}(K(\mathbb{F}_q)_p) \otimes BP_*^{\otimes j}$ for any $j \geq 0$. Thus, $d_2(\lambda_1') = 0$, and we need to check if

$$\alpha(d_2(tb)z_3 + tbd_2(z_3)) + \beta(\lambda_1' d_2(z_2)) - (t\mu)^{k-1} t\sigma b = 0$$

for any $\alpha, \beta \in \mathbb{F}_p$. Since z_3 cannot contain $t\mu$ or $t\sigma b$ as a factor, tb is not a factor of $(t\mu)^k t\sigma b$, and λ_1' is not a factor of $(t\mu)^{k-1} t\sigma b$, there are no such α and β that make this equation hold.

Step 4: We now discuss how to detect the elements β_i where $i \not\equiv 1 \mod p$. First, we will discuss how to detect β_2 in $V(1)_*T_3(K(\mathbb{F}_q)_p)$. The class β_2 is represented by $k_0 + 2b_{1,0}v_2$ mod (p, v_1) in the input of the BP-Adams spectral sequence for S. It is also a permanent cycle in the BP-Adams spectral sequence for $V(1)$ as a consequence of [26, Lemma 5.4]. It maps to the class $k_0 + 2b_{1,0}(t\mu)$ in

$$BP_*BP \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP_*V(1) \wedge T_3(K(\mathbb{F}_q)_p)$$

under the map of E_1-pages of BP-Adams spectral sequences induced by the map $V(1) \to V(1) \wedge T_3(K(\mathbb{F}_q)_p)$, by the remarks at the beginning of the proof and the multiplicativity of E_1-page of the BP-Adams spectral sequence.

Recall that $BP_*V(1) \wedge T_3(K(\mathbb{F}_q)_p)$ is isomorphic to

$$H_*\left(P(t_1^p, t_2, \dots) \otimes E(b) \otimes E(\sigma t_1^p, \sigma t_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b) \otimes P_3(t); d^2(x) = t\sigma x\right)$$

modulo d^4 differentials. We can therefore check every element in degree $4p^2 - 2p - 2$ in

$$BP_*BP \otimes_{BP_*} BP_*V(1) \wedge T_3(K(\mathbb{F}_q)_p)$$

to see if it has the element of interest as a boundary. We therefore make a table of all elements in this degree:

$$\begin{array}{cccc}
\cancel{t_1^p\otimes t_2} & v_1^i t_1^{p+1-i}\otimes \sigma b & t_1^{p+1}\otimes \sigma b & v_1^{p+1}\otimes \sigma b \\
t_1^p t_2\otimes 1 & \cancel{v_1^p\otimes t_2} & t_1^{p+1}v_1^p\otimes 1 & \cancel{v_1\otimes (t_1^p)^2} \\
t_1^p v_2\otimes 1 & v_1^p t_2\otimes 1 & t_1^{2p}v_1\otimes 1 & v_1\otimes \gamma_2(\sigma b) \\
t_1^{2p+1}\otimes 1 & v_1^p v_2\otimes 1 & \cancel{t_1\otimes (t_1^p)^2} & \cancel{v_1\otimes t_1^p\sigma b} \\
t_1^p t_2\otimes 1 & v_1^{2p+1}\otimes 1 & t_1\otimes \gamma_2(\sigma b) & \cancel{v_2\otimes t_1^p} \\
t_1^p v_1^{p+1}\otimes 1 & v_1^p t_2\otimes 1 & \cancel{t_1\otimes t_1^p\sigma b} & v_2\otimes \sigma b \\
\cancel{t_1^{p+1}\otimes t_1^p} & \cancel{v_1^{p+1}\otimes t_1^p} & t_1 v_1^{2p}\otimes 1 & v_1^i t_1^{2p+1-i}\otimes 1 \\
\cancel{t_2\otimes t_1^p} & \cancel{v_1^i t_1^{p-i}\otimes t_2} & \cancel{1\otimes t_2\sigma b} & v_1^p\otimes t\mu \\
t_2\otimes \sigma b & v_1^i t_1^{p-i}v_2\otimes 1 & t_1^p\otimes t\mu & 1\otimes \sigma b t\mu \\
\cancel{1\otimes b\sigma t_2} & v_1^i t_1^{p-i}t_2\otimes 1 & \cancel{1\otimes t_1^p t\mu} & \cancel{1\otimes (tt_1^p)\mu} \\
\cancel{1\otimes t_2 t_1^p} & \cancel{v_1^i t_1^{p+1-i}\otimes t_1^p} & v_1^i t_1^{p-i}\otimes t\mu &
\end{array}$$

where the elements that are crossed out are elements that do not survive to the E^4 page in the generalized homological homotopy fixed point spectral sequence.

We can immediately rule out any element of the form $x\otimes 1$ where $x\in BP_*BP$ because if a class of this form hit the target class then it would have also happened in the source spectral sequence. We therefore just need to check the classes:

$$\begin{array}{cccc}
v_1^i t_1^{p-i}\otimes t\mu & v_2\otimes \sigma b & t_1^p\otimes t\mu & t_1^{p+1}\otimes \sigma b \\
1\otimes \sigma b t\mu & t_1\otimes \gamma_2(\sigma b) & v_1^p\otimes t\mu & v_1^{p+1}\otimes \sigma b \\
v_1^i t_1^{p+1-i}\otimes \sigma b & v_1\otimes \gamma_2(\sigma b) & t_2\otimes \sigma b. &
\end{array}$$

We compute the d_1 differential on each of these classes:

$$\begin{array}{l}
d_1(v_1^i t_1^{p-i}\otimes t\mu) = -v_1^i \sum_{j=1}^{p-i-1}\binom{p-i-1}{j} t_1^j\otimes t_1^{p-i-j-1}\otimes t\mu \\
d_1(1\otimes \sigma b t\mu) = 1\otimes 1\otimes \sigma b t\mu \\
d_1(t_1^p\otimes t\mu) = -\sum_{i=1}^{p-1}\binom{p}{i} t_1^i\otimes t_1^{p-i}\otimes t\mu \equiv 0 \mod (p) \\
d_1(v_1^p\otimes t\mu) = v_1^p\otimes 1\otimes t\mu \\
d_1(t_2\otimes \sigma b) \equiv -t_1^p\otimes t_1\otimes \sigma b \mod (p, v_1) \\
d_1(v_2\otimes \sigma b) = v_2\otimes 1\otimes \sigma b \\
d_1(t_1\otimes \gamma_2(\sigma b)) = 0 \\
d_1(v_1\otimes \gamma_2(\sigma b)) = v_1\otimes 1\otimes \gamma_2(\sigma b) \\
d_1(t_1^{p+1}\otimes \sigma b) = -\sum_{i=1}^{p+1}\binom{p+1}{i} t_1^i\otimes t_1^{p+1-i}\otimes \sigma b \equiv t_1\otimes t_1^p\otimes \sigma b + t_1^p\otimes t_1\otimes \sigma b \mod (p) \\
d_1(v_1^{p+1}\otimes \sigma b) = v_1^{p+1}\otimes 1\otimes \sigma b \\
d_1(v_1^i t_1^{p+1-i}\otimes \sigma b) = -v_1^i \sum_{j=1}^{p+1-i}\binom{p-i+1}{j} t_1^j\otimes t_1^{p-i-j+1}\otimes \sigma b.
\end{array}$$

We observe that no linear combination these classes hits the element $k_0 + 2b_{1,0}(t\mu)$. Therefore, $k_0 + 2b_{1,0}(t\mu)$ survives to the E_2-page of the BP-Adams spectral sequence for $V(1)\wedge T_3(K(\mathbb{F}_q)_p)$.

We next need to check if it is the boundary of a d_2. However, all of the potential elements in $BP_{4p^2-4p-1}V(1)\wedge T_3(K(\mathbb{F}_q)_p)$ were killed by a d^2 differential in the generalized homological homotopy fixed point spectral sequence. Therefore, there are no elements with $k_0 + 2b_{1,0}v_2$ as a boundary.

The only classes in degree $4p^2-2p-2$ in $V(1)_*T_3(K(\mathbb{F}_q)_p)$ are $t\lambda_1'\lambda_2$ and $t\mu\sigma b$. Now, by Corollary 3.4 and the Leibniz rule, there is a differential

$$d_{2p-2}(t\mu\sigma b) = -t^p\alpha_1\mu\sigma b$$

in the generalized homological homotopy fixed point spectral sequence that computes

$$V(1)_*TC^-(K(\mathbb{F}_q)_p)$$

so the element $t\mu\sigma b$ does not survive to $V(1)_*TC^-(j)$. Therefore, $t\mu\sigma b$ cannot be in the image of the unit map

$$V(1)_*S \to V(1)_*T_3(K(\mathbb{F}_q)_p$$

because this map factors through $V(1)_*TC^-(j)$; i.e., the diagram of ring spectra

$$\begin{tikzcd}
V(1) \arrow[r] \arrow[dr] & V(1) \wedge TC^-(j) \arrow[d] \\
 & V(1) \wedge T_3(K(\mathbb{F}_q)_p)
\end{tikzcd}$$

commutes. Therefore, in degree $V(1)_{4p^2-2p-2}T_3(K(\mathbb{F}_q)_p)$ the image of the unit map is either $\mathbb{F}_p$ generated by a linear combination of $t\lambda_1'\lambda_2$ and $t\mu\sigma b$ or it is trivial in that degree. It cannot be trivial in degree $4p^2-2p-2$ because we just showed that there is a permanent cycle in the BP-Adams spectral sequence that survives to become an element in this degree in $V(1)_*T_3(K(\mathbb{F}_q)_p)$. Therefore, $V(1)_{4p^2-2p-2}T_3(K(\mathbb{F}_q)_p) \cong \mathbb{F}_p\{c\cdot t\lambda_1'\lambda_2 + c'\cdot t\mu\sigma b\}$ where $c \neq 0$. Since

$$0 = d_{2p-2}(c\cdot t\lambda_1'\lambda_2 + c'\cdot t\mu\sigma b) = c\cdot d_{2p-2}(t\lambda_1'\lambda_2) + c'\cdot d_{2p-2}(t\mu\sigma b) = c\cdot d_{2p-2}(t\lambda_1'\lambda_2) - c'\cdot t^p\alpha_1\mu\sigma b$$

where $c, c' \in \mathbb{F}_p$ and $c \neq 0$. We see that

$$d_{2p-2}(t\lambda_1'\lambda_2) = c^{-1}\cdot c'\cdot t^p\alpha_1\mu\sigma b$$

and therefore $c' \neq 0$. Since we already identified that $t\mu$ is a permanent cycle and $b_{1,0}$ is a permanent cycle that represents the homotopy class $t\sigma b$, we know $2b_{1,0}t\mu$ is represents $2t\sigma bt\mu$ if it survives. Conequently, $c = 1$ and $c' = 2$ (up to multiplying each of these by the same unit).

Step 5: We now discuss how to detect β_k where $k \not\equiv 1 \mod p$. In this case, β_k is represented by $\binom{k}{2}v_2^{k-1}k_0 + kb_{1,0}v_2^k$, which maps to $(t\mu)^{k-1}(\binom{k}{2}k_0 + kb_{1,0}(t\mu))$ so we just need to check the d_1 differentials on classes of the form $(t\mu)^{k-1}w$ where w is an element in $BP_*BP\otimes_{BP_*}BP_*BP$. If the class $(t\mu_2)^k(2t_1^p\otimes t_2\otimes 1 - 2t_1^p\otimes t_1^{1+p}\otimes 1 - t_1^{2p}\otimes t_1\otimes 1)$ is the co-boundary of a d_1, then there is a sum of classes $\sum_i a_i\otimes m_i \in BP_*BP\otimes_{BP_*}BP_*V(1)\wedge T_k(K(\mathbb{F}_q)_p)$ such that $d_1(\sum_i a_i\otimes m_i)\doteq(t\mu_2)^k(2t_1^p\otimes t_2\otimes 1 - 2t_1^p\otimes t_1^{1+p}\otimes 1 - t_1^{2p}\otimes t_1\otimes 1)$.

Sub-step 1: Recall that the co-action on m is of the form $\psi(m) = 1\otimes m + \sum_j a_j\otimes m_j$ where $|m_j| < |m|$. Again, observe that the only elements in $(BP\wedge V(1))_*T_k(K(\mathbb{F}_q)_p)$ whose co-action contains $(t\mu)^{k-1}$ as either m or m_j for some j are classes of the form $(t\mu)^{k-1}y$ for some $y \in (BP\wedge V(1))_*T_k(K(\mathbb{F}_q)_p)$ up to multiplication by a unit. The co-action of such a class is $\psi((t\mu)^{k-1}y) = (1\otimes(t\mu)^{k-1})\psi(y)$, and $\psi(y)$ must be of the form $\psi(y) = 1\otimes y + z\otimes 1 + \sum b_i\otimes y_i$ since $\psi((t\mu)^{k-1}y)$ must have $1\otimes(t\mu)^{k-1}$ as a term, up to multiplication by a unit. Since the only classes in $(BP\wedge V(1))_*T_k(K(\mathbb{F}_q)_p)$ that have a term $z\otimes 1$ in their co-action are the classes t_1^p, t_i for $i \geq 2$ the class y must be a product of these. Since $|(u\cdot t\mu)^{k-1}y| = (2p^2-2)(k-1)+|y|$ and the degree must equal $4p^2-2p+k(2p^2-2)$, the degree of y must be $4p^2-2p$. However, the class t_1^p is the element of lowest degree in the set $\{t_1^p, t_2, \dots\}$ and $|t_1^p| = 2p^2-2p$ and the next lowest degree element is t_2 with $|t_2| = 2p^2-2$ so no product of classes in this set can be in degree $4p^2-2p$. Thus, $m_i\doteq(t\mu)^{k-1}$ for at least one i.

Now, if $m_i\cdot = (t\mu)^{k-1}$ for only one i, then the element a_i corresponding to m_i must have reduced co-product $2t_1^p\otimes t_2\otimes 1 - 2t_1^p\otimes t_1^{1+p}\otimes 1 - t_1^{2p}\otimes t_1\otimes 1 + z$ for some class z in

$BP_*BP \otimes_{BP_*} BP_*BP$; i.e

$$\bar{\Delta}(a_i)\dot{=}(2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p} \otimes t_1 \otimes 1).$$

The degree of a_i must be $4p^2 - 2p$, so $a_i \dot{=} t_1^j v_1^{p-j} v_2^{\epsilon_1} t_2^{\epsilon_2}$.

However,

$$\Delta(t_1^j v_1^{p-j} v_2^{\epsilon_1} t_2^{\epsilon_2}) \quad = \quad v_1^{p-j} v_2^{\epsilon_1}(t_1 \otimes 1 + 1 \otimes t_1)^j (t_2 \otimes 1 + 1 \otimes t_2 + t_1^p \otimes t_1)^{\epsilon_2}$$

and so $\bar{\Delta}(t_1^j v_1^{p-j} v_2^{\epsilon_1} t_2^{\epsilon_2})$ does not equal

$$t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p} \otimes t_1 \otimes 1 + z$$

up to multiplication by a unit for any j, and any element $z \in BP_*BP \otimes_{BP_*} BP_*BP$.

Suppose that $m_i = (t\mu)^{k-1}$ for $i \in I$ where I contains more than one natural number. Then

$$\psi(\sum_{i \in I} a_i) \cdot = 2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p} \otimes t_1 \otimes 1 + z'$$

for some possibly trivial element z' in $BP_*BP \otimes_{BP_*} BP_*BP$. However, we checked in Step 4 that no class of the form $\sum_{i \in I} a_i \otimes 1$ has co-action

$$(2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p} \otimes t_1 \otimes 1) + z'$$

and the same proof applies here.

Thus, there is no sum of classes $\sum_i a_i \otimes m_i$ such that

$$d_1(\sum_i a_i \otimes m_i) = (2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p}) \otimes (t\mu)^{k-1}$$

and therefore the class $(2t_1^p \otimes t_2 \otimes 1 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p}) \otimes (t\mu)^{k-1}$ survives to the E_2-page.

Sub-step 2: To see that there are no d_2 differentials that hit $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ we need to check that no elements in $BP_{(2p^2-2)k+4p^2-2p-1}V(1) \wedge F(S(\mathbb{C}^k)_+, THH(j))^{\mathbb{T}}$ for $k \geq 2$ have $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ as a boundary. The only possible classes are elements in one of the families

$$\{(\lambda_1' t_1^{p^2-p})z_1, (\lambda_2 t_2^{p-1})z_2, (b\gamma_{p-1}(\sigma b))z_3\}$$

where $z_i \in P(t_1^{p^2}, t_2^p, t_3, \dots) \otimes E(\lambda_1' t_1^{p^2-p}, \lambda_2 t_2^{p-1}, b\gamma_{p-1}(\sigma b)) \otimes P(\mu_2) \otimes \Gamma(\gamma_p(\sigma b)) \otimes P(t)/t^{k+1}$ for $i = 0, 1, 2$, or elements in one of the families

$$\{\lambda_1' y_1, \lambda_2 y_2, (t^k b)y_3, \}$$

where $y_i \in P(t_1^p, t_2, t_3, \dots) \otimes E(\lambda_1', \lambda_2, b) \otimes P(\mu_2) \otimes \Gamma(\sigma b)$ for $i = 0, 1, 2$ by Corollary 3.9. There are differentials

$$d_1(\lambda_1' t_1^{p^2-p}) = -t_1^{p^2-p} \otimes \lambda_1'$$

and

$$d_1(\lambda_2 t_2^{p-1}) = (t_1 \otimes \lambda_1')d_1(t_2^{p-1})$$

and $d_1(t_2^{p-1}) \neq 0$, so these classes do not survive to the E^2 page and therefore no class in the families $\lambda_1' y_1$ and $\lambda_2 y_2$ can be an element in the E_2-page that has $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ as a boundary. We know $d_1(b\gamma_{p-1}(\sigma b)) = 0$, so in order for $b\gamma_{p-1}(\sigma b)z_3$ to survive to the E_2-page, z_3 must be a comodule primitive so that $d_1(z_3) = 0$ as well and hence by the Leibniz rule $d_1(b\gamma_{p-1}(\sigma b)z_3) = 0$. The only comodule primitives are products of elements in the set $\{\mu_2, \gamma_{p^k}(\sigma b), t^k \mid k \geq 1\}$. Since $|b\gamma_{p-1}(\sigma b)| = 2p^3 - 2p^2 - 1$ and the homotopy degree of

$\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ is $(2p^2-2)k + 2p^2 - 2p - 2$, then we must have

$$|b\gamma_{p-1}(\sigma b)z_3| = (2p^2-2)k + 2p^2 - 2p - 2 + 1$$

so $|z_3| = (2p^2-2)k+2p^2-2p-2+1-2p^3+2p^2+1$. However, $|\mu_2| \equiv 0 \mod 2p^2$, $|\gamma_{p^k}(\sigma b)| \equiv 0 \mod 2p^2$, $t^k \equiv -2k \mod 2p^2$, and $|z_3| \equiv -2k-2p \mod 2p^2$, so no product of elements in the set $\{\mu_2, \gamma_{p^k}(\sigma b), t^k \mid k \geq 1\}$ can be congruent to $|z_3|$ modulo $2p^2$ (note that we use the fact that $(t^k)^2 = 0$ here). Thus, there is no element that is both a comodule primitive and in the correct degree, so element of the form $b\gamma_{p-1}(\sigma b)z_3$ can have $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ as a boundary. We now consider the elements in the second set of families of elements. Notice that in the second set of families, none of the elements y_i have t as a factor. First, we note that $d_1(\lambda_2) = t_1 \otimes \lambda_1'$ so $\lambda_2 y_2$ does not survive to the E_2 page and therefore it cannot have $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ as a boundary. The elements $t^k b$ and λ_1' are comodule primitives, so they survive to E_2. However, as we discussed before in order for $\lambda_1' y_1$ and $t^k b y_3$ to survive to E_2 as well, then by the Leibniz rule y_1 and y_3 must be comodule primitives. $|\lambda_1'| = 2p^2-2p+1$, and we know

$$|\lambda_1' y_1| = (2p^2-2)k + 2p^2 - 2p - 2 + 1,$$

which implies $|y_1| = (2p^2-2)k + 2p^2 - 2p - 2 + 1 - (2p^2 - 2p + 1) = (2p^2-2)k - 2$. Note that the only comodule primitives that y_1 could be are products of elements in the set $\{\mu_2, \gamma_{p^j}(\sigma b) \mid j \geq 0\}$. We know $|\mu_2| \equiv 0 \mod 2p^2$ and $|\gamma_{p^j}(\sigma b)| \equiv 0 \mod 2p^2$ for $j \geq 0$. Since $|y_1| \equiv -2(k+1)$, the only way that a product of one of these classes could have the correct degree is if $k+1 \equiv 0 \mod p^2$. However, that would imply that $k = p^2\ell + 1$ for some integer ℓ. In that case, $\binom{\ell p^2+1}{2}(t\mu)^{\ell p^2}k_0 + (p^2\ell+1)b_{1,0}(t\mu)^{\ell p^2+1} = b_{1,0}(t\mu)^{pm+1}$ for $m = p\ell$, since $\binom{\ell p^2+1}{2} = (p^2+1)(p^2)/2 \equiv 0 \mod p$, in which case we already proved that $b_{1,0}(t\mu)^{pm+1}$ is a permanent cycle for $m \geq 1$ an integer. Therefore, this does not occur. The last case to consider is the family of classes $t^k b y_3$. In this case,

$$|t^k b y_3| = (2p^2-2)k + 2p^2 - 2p - 2 + 1$$

implies that $|y_3| = (2p^2-2)k+2p^2-2p-2+1-(-2k+2p^2-2p-1) = 2p^2k-2$. Again, the only comodule primitives in even degrees are products of elements in the set $\{\mu_2, \gamma_{p^j}(\sigma b) \mid j \geq 0\}$. We observe that $|y_3| \equiv -2 \mod 2p^2$, whereas $|\mu_2| \equiv 0 \mod 2p^2$, and $|\gamma_{p^j}(\sigma b)| \equiv 0 \mod 2p^2$ so no such element y_3 exists such that $d_1(t^k b y_3) \neq 0$. Thus, there is no possible class in the correct degree at the E_2-page, which could have $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ as a boundary when $k \not\equiv 1 \mod p$, which covers all the remaining cases.

We now just need to show that the class in $V(1)_{(2p^2-2)k-2}T_k(K(\mathbb{F}_q)_p)$ is represented by $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ is in fact $\binom{k}{2}(t\mu)^{k-1}(t\lambda_1'\lambda_2) + k(t\sigma b)(t\mu)^k$. To see this, note that

$$\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k = \binom{k}{2}(t\mu)^{k-2}\left(k_0 + 2b_{1,0}(t\mu)\right) + (2k-k^2)(b_{1,0})(t\mu)$$

for $k \geq 1$ so since $k_0 + 2b_{1,0}(t\mu)$ survives to become $t\lambda_1'\lambda_2 + 2(t\sigma b)(t\mu)$, $t\mu$ survives to become $t\mu$ and $b_{1,0}$ survives to become $t\sigma b$, we see that $\binom{k}{2}(t\mu)^{k-1}k_0 + kb_{1,0}(t\mu)^k$ survives to become $\binom{k}{2}(t\mu)^{k-1}(t\lambda_1'\lambda_2) + k(t\sigma b)(t\mu)^k$. □

3.4. **Detecting the β-family in iterated algebraic K-theory.** The goal of this section is to prove that the β-family is detected in the iterated algebraic K-theory of finite fields. We prove this as a Corollary to Theorem 10. The proof relies on the fact that the trace map $K(R) \to TC^-(R)$ is a map of commutative ring spectra when R is a commutative ring spectrum. The proof that the cyclotomic trace map $K(R) \to TC(R)$ is a map of commutative ring spectra when R is a commutative ring spectrum is due to Hesselholt-Geisser [16] for Eilenbrg-MacLane spectra and later Dundas [15] and Blumberg-Gepner-Tabuada [11] for commutative ring spectra. The advantage of the approach of Blumberg-Gepner-Tabuada [11] is that they prove that the multiplicative cyclotomic trace map is also unique. This work builds on their proof that algebraic K-theory is the universal additive functor [10] (also see Barwick [8]).

We believe the fact that the trace map $K \to TC^-$ is multiplicative is well known, but it is not explicitly stated in the literature to our knowledge, so we include a proof. Note that by the Nikolaus-Scholze equalizer [25, Cor. 1.5], there is a natural transformation $TC \to TC^-$ and therefore there exists a natural transformation $K \to TC^-$. The proof will then follow from the fact that algebraic K-theory is initial amongst multiplicative additive functors by [11, Cor. 7.2].

Lemma 3.13. Suppose R is a commutative ring spectrum, then the trace map $K(R) \to TC^-(R)$, which factors through $TC(R)$, is a map of commutative ring spectra.

Proof. By [11, Cor. 7.2], it suffices to show that $TC^-(-)$ is an E_∞-object in the symmetric monoidal ∞-category of additive functors from $\mathrm{Cat}_\infty^{\mathrm{perf}}$, the ∞-category of small idempotent-complete stable infinity categories and exact functors, to the stable ∞-category of spectra, denoted $\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathrm{S}_\infty)^{\otimes}$. Since E_∞-objects in a functor category of infinity stable categories are equivalent to commutative monoids in this functor category and by [17, Prop. 2.12] [14, Ex. 3.2.2] an E_∞-object in this functor category is equivalent to a lax symmetric monoidal functor. The fact that the functor TC^- is lax symmetric monoidal follows by the diagram

$$\begin{tikzcd}
F(E\mathbb{T}_+, THH(-))^{\mathbb{T}} \wedge F(E\mathbb{T}_+, THH(-))^{\mathbb{T}} \arrow[d, "\wedge"] \\
F((E\mathbb{T} \times E\mathbb{T})_+, THH(-) \wedge THH(-))^{\mathbb{T}}) \arrow[d, "\mu_\# \Delta_+^\#"] \\
F(E\mathbb{T}_+, THH(-))^{\mathbb{T}})
\end{tikzcd}$$

where $\Delta_+^\#$ is induced by the diagonal $\Delta_+ \colon E\mathbb{T}_+ \to (E\mathbb{T} \times E\mathbb{T})_+$ and $\mu_\#$ is induced by the E_∞-structure of THH as an -object in $\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathrm{S}_\infty)^{\otimes}$ (cf. [13, Sec. 4]). Specifically,

$$\Delta_+^\# = F(\Delta_+, THH(-) \wedge THH(-))^{\mathbb{T}}$$

and $\mu_\# = F(E\mathbb{T}_+, \mu)^{\mathbb{T}}$ where $\mu \colon THH(-) \wedge THH(-) \to THH(-)$ is the multiplication map of the E_∞-object THH in $\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathrm{S}_\infty)^{\otimes}$. The fact that $TC^-(-)$ is an additive functor follows from [19, 25], or see [32, Sec. 2.2.13]. □

Remark 3.14. The proof above implies by [11, Cor. 7.2] that there is a unique morphism from $K \to TC^-$ in the ∞-category of E_∞-objects in $\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathrm{S}_\infty)^{\otimes}$, which is stronger than what is needed for the statement of the lemma. We give the simpler statement because that is the version that will be used in the next proof. In the end, the result will be a result about homotopy groups, so the change of model from the model category of symmetric spectra $\mathfrak{S}$ to the infinity category of spectra S_∞ should not be concerning.

Corollary 3.15. Let $p \geq 5$ be a prime and q be a prime power that generates $(\mathbb{Z}/p^2)^\times$. The classes β_i map from $\pi_* S$ to nonzero elements in $\pi_* K(K(\mathbb{F}_q))$ under the unit map.

Proof. First, the classes β_i in $V(1)_*$ map to $V(1)_*K(K(\mathbb{F}_q)_p)_p$ under the unit map since the cyclotomic trace is multiplicative and therefore the map

$$V(1)_*S \to V(1)_*TC^-(K(\mathbb{F}_q)_p)$$

factors through $V(1)_*K(K(\mathbb{F}_q)_p)$; i.e, there is a commutative diagram

$$\begin{tikzcd} V(1)_*S \arrow[r, "V(1)_*\eta_{K(K(\mathbb{F}_q)_p)}"] \arrow[dr, "V(1)_*\eta_{TC^-(K(\mathbb{F}_q)_p)}"'] & V(1)_*K(K(\mathbb{F}_q)_p) \arrow[d, "V(1)_*tr"] \\ & V(1)_*TC^-(K(\mathbb{F}_q)_p). \end{tikzcd}$$

There is also a commuting diagram of ring spectra

$$\begin{tikzcd} S \simeq S \wedge S \arrow[r, "1_S\wedge\eta"] \arrow[d, "i_0i_1\wedge 1_S"'] & S \wedge K(K(\mathbb{F}_q)) \arrow[d, "i_0i_1\wedge 1_{K(K(\mathbb{F}_q))}"'] \arrow[dr, "i_0i_1\wedge K(f_p)"] & \\ V(1)\wedge S \arrow[r, "1_{V(1)}\wedge\eta"] & V(1)\wedge K(K(\mathbb{F}_q)) \arrow[r, "1_{V(1)}\wedge K(f_p)"] & V(1)\wedge K(K(\mathbb{F}_q)_p) \end{tikzcd}$$

where $f_p : K(\mathbb{F}_q) \to K(\mathbb{F}_q)_p$ is the p-completion map and η is the unit map. Since the classes β_i pull back to π_*S along the unit map and since they map nontrivially to classes in $\pi_*V(1)\wedge K(K(\mathbb{F}_q)_p)$, they must map to nontrivial classes in $\pi_*K(K(\mathbb{F}_q))$ under the unit map

$$\pi_*S \to \pi_*K(K(\mathbb{F}_q)).$$

□

Corollary 3.16. Let $\mathcal{O}_F$ be the ring of integers in a number field F whose residue field is $\mathbb{F}_q$ for some prime power q which generates $(\mathbb{Z}/p^2\mathbb{Z})^\times$. Then the β-family is detected in $K(K(\mathcal{O}_F))$. In particular, the β-family is detected in $K(K(\mathbb{Z}))$.

Proof. Let F be a number field and q a prime power satisfying the conditions in the statement of the corollary. Since $\mathcal{O}_F$ has residue field $\mathbb{F}_q$, there exists a map of commutative rings $\mathcal{O}_F \to \mathbb{F}_q$ inducing a map of commutative ring spectra $K(K(\mathcal{O}_F)) \to K(K(\mathbb{F}_q))$. Therefore, there is a commutative diagram

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & K(K(\mathbb{F}_q)) \\ & K(K(\mathcal{O}_F)) \arrow[ur] & \end{tikzcd} \tag{12}$$

of commutative ring spectra. Since the β-family is nontrivial in the image of the unit map $\pi_*S \to \pi_*K(K(\mathbb{F}_q))$, it is also nontrivial in the image of the unit map $\pi_*S \to \pi_*(K(K(\mathcal{O}_F)))$. In particular, let $p = 5$, then $q = 2$ generates $(\mathbb{Z}/25\mathbb{Z})^\times$ and consequently it topologically generates $\mathbb{Z}_5^\times$. Thus, there is a map of commutative rings $\mathbb{Z} \to \mathbb{F}_2$ inducing a map of commutative ring spectra $K(K(\mathbb{Z})) \to K(K(\mathbb{F}_2))$. Since the β-family is detected in $K(K(\mathbb{F}_2))$, by the same diagram (12) with $\mathcal{O}_F = \mathbb{Z}$ we see that the β-family is detected in iterated algebraic K-theory of the integers. □

Note that the α-family is detected in $K(\mathbb{Z})$. Since $K_0(\mathbb{Z}) \cong \mathbb{Z}$, there is a map of commutative ring spectra $K(\mathbb{Z}) \to H\mathbb{Z}$. We may consider the infinite family of maps

$$S \to \ldots \to K(K(K(\mathbb{Z}))) \to K(K(\mathbb{Z})) \to K(\mathbb{Z})$$

and a specialization of the Greek-letter family red-shift conjecture is that the n-th Greek letter family is in the image of the unit map $S \to K^{(n)}(\mathbb{Z})$ where $K^{(n)}(\mathbb{Z})$ is algebraic K-theory iterated n-times. As a consequence of Corollary 3.16, we have therefore proved this version of the conjecture for $n = 2$.

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