Lecture 2:

The Whitehead group

I. Motivation

In the 1950's, Whitehead studied the "simple homotopy type" of a finite CW complex.

Q: Given a htpy equivalence $X \xrightarrow{\simeq} Y$ between finite CW complexes, when is X simple homotopy equivalent to Y; i.e when can we write a homotopy between X and Y in terms of elementary expansions and collapses.

More precisely, let (K, L) be a finite CW pair. Then $K \overset{e}{\searrow} L$ "K collapses to L via an elementary collapse" if

1) $$K = L \cup e^{n-1} \cup e^n$$

where $e^{n-1}, e^n \notin L$

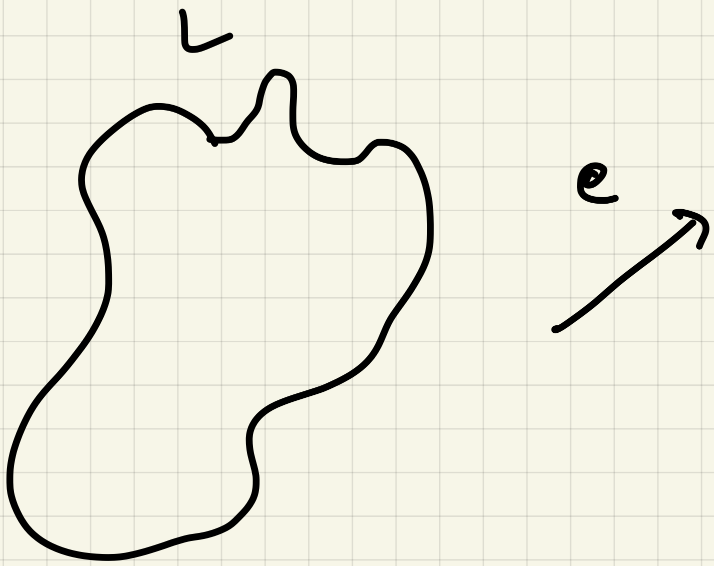
2) there exists a pair (D^n, D^{n-1}) and a map $\varphi : D^n \longrightarrow K$ such that

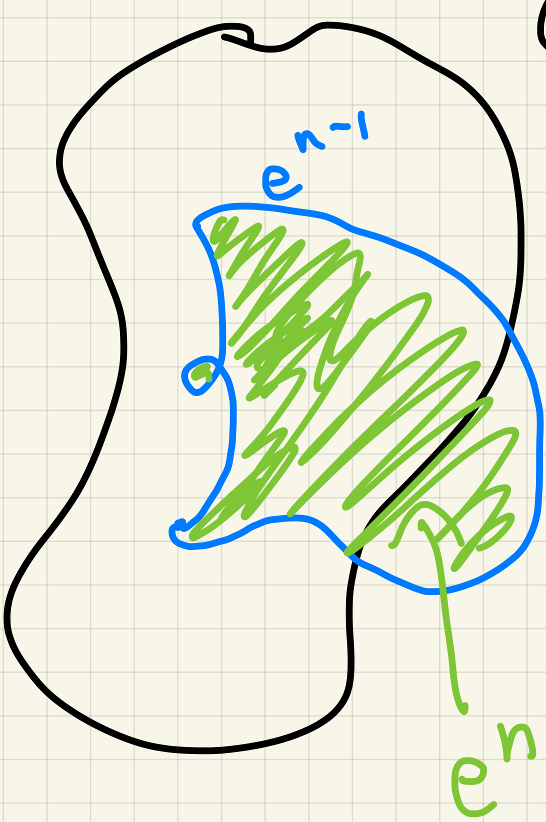
$$\begin{tikzcd}
\partial D^{n-1} \arrow[r] \arrow[d, hook] & L \arrow[d] \\
\varphi|_{D^{n-1}} : D^{n-1} \arrow[r] & L \cup e^{n-1}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d] & L \cup e^{n-1} \arrow[d] \\
\varphi : D^n \arrow[r] & L \cup e^{n-1} \cup e^n
\end{tikzcd}$$

$$\varphi(cl(\partial D^{n-1} - D^n)) \subseteq L^{n-1}$$

We also write $L \overset{e}{\nearrow} K$ and say "L expands to K via an elem. expansion".

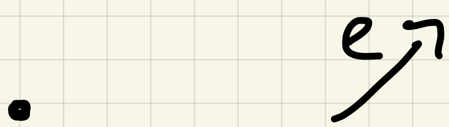
Picture:

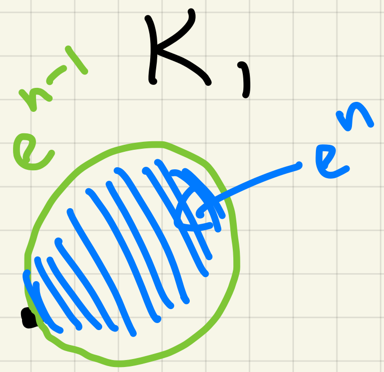




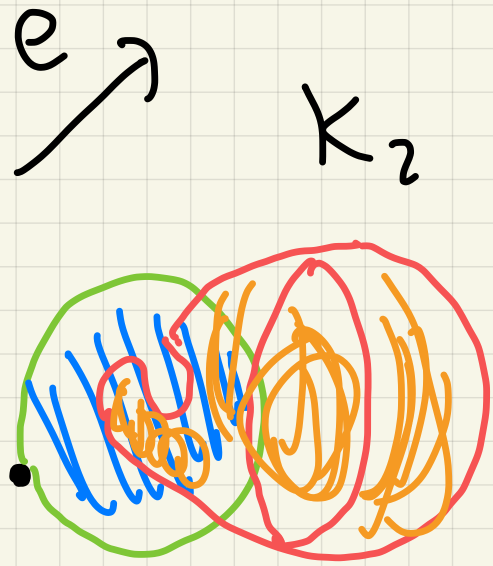
Example:

$L = L^0$





L is simple K_2



Example: Lens spaces

$$L(p,q) = \frac{\{(z_1,z_2)\in \mathbb{C}^2 \mid |z_1|^2+|z_2|^2=1\}}{(z_1,z_2)\sim(\zeta z_1, \zeta^q z_2)}$$

$\zeta = e^{2\pi i/p}$ (p-th root of unity)

$$L(2,1) = \mathbb{R}P^3$$

There exists a homotopy equivalence

$$f : L(7,2) \xrightarrow{\simeq} L(7,1),$$

which is **not** a simple homotopy equivalence.

(See Exercise on p. 98 "A course in simple homotopy theory" M.M. Cohen 1972)

II. Whitehead group

Let R be a ring.

$$GL_n(R) = \{ n \times n \text{ invertible matrices} \}$$

$$GL_n(R) \hookrightarrow GL_{n+1}(R)$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

$$GL(R) := \operatorname*{colim}_n GL_n(R)$$

Def:

$$K_1(R) := GL(R)^{ab}$$
$$:= GL(R) / [GL(R), GL(R)]$$

Comutator subgroup $\rightarrow [GL(R), GL(R)]$.

<u>Def:</u> A transvection $e_{ij}(\lambda)$ in $GL_n(R)$ where $\lambda \in R$, $1 \leq i \neq j \leq n$ is a matrix of the form

$$\begin{array}{cccc} & & & \\ \left(\begin{array}{cccc} 1 & 0 & & 0 \\ & 1 & 0 \;\; \lambda & \\ & & \ddots & 0 \\ 0 & & & 1 \end{array}\right) & j \\ \quad\quad\quad i & \end{array} .$$

These are elementary matrices and so we write

$$E_n(R) \subseteq GL_n(R)$$

for the subgroup of $GL_n(R)$ generated by the transvections.

Again,

$$E_n(R) \hookrightarrow E_{n+1}(R)$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and we define

$$E(R) = \operatorname{colim}_n E_n(R).$$

Def: we say a group G is perfect if

$G = [G,G]$. In this case $G/[G,G] = 1$.

Rmk: If $\pi_1 X \neq 1$ is perfect and $\pi_n X = 0$ $n \neq 1$, then

$$\tilde{H}_*(X) = 0 \text{ but } X \not\simeq *.$$

Lemma 1: Let $n \geq 3$. Then

$E_n(R)$ is perfect.

Pf: $e_{ij}(\lambda) = [e_{ik}(\lambda), e_{kj}(1)]$. $\lambda \in R$

$i \neq j \neq k$. (Ex: $E_2(\mathbb{F}_2)$ is not perfect)

Lemma 2: (Whitehead's lemma)

$$E(R) = [GL(R), GL(R)].$$

Proof: By lemma 1

$$E(R) \subseteq [GL(R), GL(R)].$$

We can write any commutator of $g, h \in GL_n(R)$ as

$$[g,h] = \underbrace{\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}}_{\in \, GL_{2n}(R)}.$$

Any matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

for $A \in GL_n(R)$ is in $E_{2n}(R)$ (Exercise). $\square$

Def: When R is commutative, there is a map

$$K_1(R) \xrightarrow{\det} \underbrace{R^\times}_{\text{units in } R} = GL_1(R)$$

and we write

$$\left(SL_n(R) \to SL_{n+1}(R), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$SK_1(R) := \ker K_1(R).$$

$$SK_1(R) = SL(R) / E(R) \qquad SL(R) = \operatorname{colim} SL_n(R).$$

$$
\begin{tikzcd}
E_n(R) \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "A \mapsto 1"] & GL_n(R) \arrow[r, "\det"] \arrow[d, hook] & R^\times \\
E_{n+1}(R) \arrow[r] \arrow[d, hook] & GL_{n+1}(R) \arrow[ur, "\det"] \arrow[d, hook] & \\
\vdots & \vdots & \\
E(R) \arrow[r, hook] \arrow[uuurr, bend right, "A \mapsto 1"', "\forall"] & GL(R) \arrow[uuur, bend right, "\det"'] \arrow[dr] & \\
& & GL(R)/E(R) \arrow[uuuu, bend right, "\det"']
\end{tikzcd}
$$

Example: When $R = \mathbb{Z}$

$$K_1(\mathbb{Z}) \cong \mathbb{Z}^\times \cong \mathbb{Z}/2 \quad \text{(Exercise)}$$

and

$$SK_1(\mathbb{Z}) = 0$$

Example: (Example 1.5.3 K book)

When $R = \mathbb{R}^{S^1}$ (continuous maps $S^1 \to \mathbb{R}$)

$$SK_1(\mathbb{R}^{S^1}) \cong \mathbb{Z}/2 \quad \text{and}$$

$$K_1(\mathbb{R}^{S^1}) = (\mathbb{R}^{S^1})^\times \oplus \mathbb{Z}/2$$

Definition:

$\langle S \rangle$ "subgroup generated by S."

$$Wh_1(G) = \frac{K_1(\mathbb{Z}[G])}{\langle \pm g \mid g \in G \rangle}$$

$\pm g \in G$ is in $GL_1(\mathbb{Z}(G)) = \mathbb{Z}[G]^\times$

III. Applications

<u>Thm</u>: (Whitehead) Suppose

$K \xrightarrow{f} L$ is a homotopy

equivalence of finite cw

complexes w/ $G = \pi_1(K) \cong \pi_1(L)$

then there is a class

$$\tau(f) \in Wh_1(G)$$

called the Whitehead torsion

of f s.t. $\tau(f) = 0$ iff f

is a simple htpy equivalence.

A triple (W, M, N) of PL-manifolds is said to be an h-cobordism if

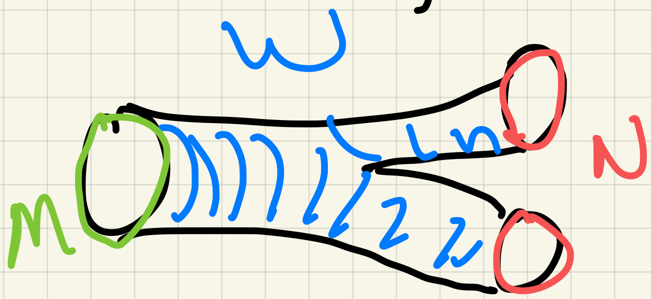
$\partial W = M \amalg N$ and

$$M \overset{\simeq}{\hookrightarrow} W \overset{\simeq}{\hookleftarrow} N$$

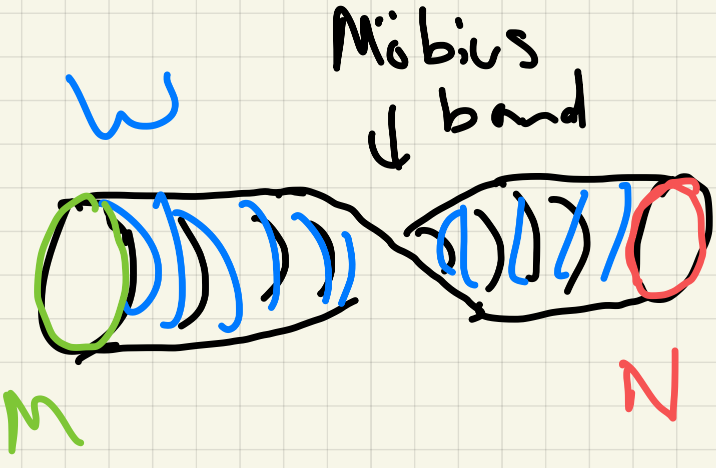
← homotopy equivalences

Then $\exists\ \tau \in Wh_1(\pi_1(M))$.

Example:



is a cobordism from S^1 to $S^1 \amalg S^1$ but <u>not</u> an h-cobordism



is an h-cobordism from S^1 to S^1

(Mazur, Smale, ...)

<u>Thm</u> [s-cobordism Thm]

PL h-cobordism $(W, M, M') \cong (M \times [0,1], M \times 0, M \times 1)$ PL homeomorphism

iff $\tau = 0$.

Moreover,

every elt. $\tau \in Wh_1(\pi_1 M)$

is the torsion of some

h-cobordism

(W, M, M').

<u>Corollary</u> (Smale)
(Generalized Poincare conjecture)

Let N be an n dimensional PL manifold with $N \simeq S^n$ for $n \geq 5$.

Then $N \underset{PL}{\cong} S^n$

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Pf: Let

$W = N - (D_1 \amalg D_2)$. Then

$(W, S_1^{n-1}, S_2^{n-1})$ is an h-cobordism.

$\pi_1(S_1^{n-1}) = 0$ so by

the s-cobordism theorem

$$(W, S_1^{n-1}, S_2^{n-1}) \overset{PL}{\cong} (S^{n-1} \times [0,1], S_1^{n-1}, S_1^{n-1})$$

So consequently

$$N = W \cup (D_1 \amalg D_2) \overset{PL}{\cong} S^{n-1} \times [0,1] \cup (D_1 \amalg D_2)$$

$$\overset{PL}{\cong} S^n.$$

III. Relating K_0 and K_1

Let $I \subset R$ a (two sided) ideal in a ring R.

Def:

$$GL(I) = \ker\left(GL(R) \to GL(R/I)\right)$$

Rmk: This definition turns out to be independent of R in the sense that if $R \to S$ is a map of rings and I maps isomorphically onto I in S then

$$\begin{tikzcd} GL(I) \arrow[r] \arrow[d, "\wr\wr"] & GL(R) \arrow[r] \arrow[d] & GL(R/I) \arrow[d] \\ GL(I) \arrow[r] & GL(S) \arrow[r] & GL(S/I) \end{tikzcd}$$

.

Def:

$E_n(R,I)$ is the normal subgroup of $GL(I)$ generated by matrices

$$e_{ij}(r)$$

such that $r \in I$ and $1 \leq i \neq j \leq n$.

Let
$$E(R,I) = \operatorname{colim}_n E_n(R,I).$$

Lemma: (Relative Whitehead Lemma)

$E(R,I) \triangleleft GL(I)$ and

$$E(R,I) = [GL(I), GL(I)].$$

(Proof is very similar to the Whitehead lemma.)

Def: Let $I \subset R$ be a two-sided ideal. Then we define $R \oplus I$ to be the ring, whose underlying abelian group is $R \oplus I$ w/ multiplication

$$
\begin{tikzcd}
(R \oplus I) \otimes (R \oplus I) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & R \oplus I \\
R \otimes R \oplus R \otimes I \oplus I \otimes R \oplus I \otimes I \arrow[ur] &
\end{tikzcd}
$$

$$R \otimes R \xrightarrow{\mu_R} R \hookrightarrow R \oplus I$$

$$R \otimes I \xrightarrow{\psi_R} I \hookrightarrow R \oplus I$$

$$I \otimes R \xrightarrow{\psi_L} I \hookrightarrow R \oplus I$$

$$I \otimes I \xrightarrow{0} I \hookrightarrow R \oplus I$$

Def:

$$K_1(R,I) := GL(I)/E(R,I)$$

$$K_0(I) := \ker\big(K_0(R \oplus I) \to K_0(R)\big)$$

$$\parallel$$

$K_0(R,I)$ where $R \oplus I$ is the square zero extension of R by I.

Note: $E(R,I)$ depends on R, so $K_1(R,I)$ depends on R whereas $K_0(I)$ does not.

$q : R \to R/I$

Proposition: There is an exact sequence

$$
\begin{tikzcd}
K_1(R,I) \arrow[r, "\text{not nec. injective}"'] & K_1(R) \arrow[r, "K_1(q)"] & K_1(R/I) \arrow[dll, "\partial"] \\
K_0(I) \arrow[r] & K_0(R) \arrow[r, "K_0(q)"] & K_0(R/I).
\end{tikzcd}
$$

Proof: I will leave it as an exercise to show that there is an exact sequence

$$
\begin{tikzcd}
1 \arrow[r] & GL(I) \arrow[r, "\text{injective}"] & GL(R) \arrow[r, "GL(q)"] & GL(R/I) \arrow[dll, "\partial_0"] \\
 & K_0(I) \arrow[r] & K_0(R) \arrow[r, "K_0(q)"] & K_0(R/I).
\end{tikzcd}
$$

Assuming this, we just need to show exactness at $K_1(R/I)$ and $K_1(R)$.

$$
\begin{tikzcd}
K_0(I) \arrow[r] \arrow[d, "\cong"] & K_0(R \oplus I) \arrow[r] \arrow[d] & K_0(R) \arrow[d] \\
\ker(K_0(q)) \arrow[r] & K_0(R) \arrow[r] & K_0(R/I)
\end{tikzcd}
$$

affine schemes

$$
\begin{tikzcd}
\mathrm{Spec}(R/I) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spec}(R) \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(R \oplus I)
\end{tikzcd}
$$

(Milnor square

$$
\begin{tikzcd}
R \oplus I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R \arrow[d] \\
R \arrow[r] & R/I
\end{tikzcd}
$$

)

Passing to quotients, we have

$$\begin{tikzcd} GL(R) \arrow[r, "GL(q)"] \arrow[d, "can_R"] & GL(R/I) \arrow[d, "can_{R/I}"] \arrow[dr, "\partial_0"] & \\ GL(R)/E(R) \arrow[r, dashed, "K_1(q)"] & GL(R/I)/E(R/I) \arrow[r, dashed, "\partial"] & K_0(I) \end{tikzcd}$$

where $\dashrightarrow$ exists by the univeral

property of abelianization

$$GL(A)/E(A) = GL(A)^{ab}$$

for $A = R, R/I$.

The kernel $\ker(\partial)$ satisfies

$$\ker(\partial) = \ker(\partial_0) \bmod E(R/I)$$

The image $\operatorname{im} K_1(q) = \operatorname{im}(can_R \circ K_1 q)$

$= \operatorname{im}(GL(q) \circ can_{R/I})$

$= \ker(\partial_0) \bmod E(R/I).$

$= \ker(\partial)$

It therefore suffices to show that the sequence is exact at $K_1(R)$. Let

$$g \in \ker\left(GL(R) \overset{can_R}{\twoheadrightarrow} K_1(R) \xrightarrow{K_1(q)} K_0(R/I)\right)$$

then by commutativity of

$$\begin{tikzcd}
GL(R) \arrow[r, "GL(q)"] \arrow[d, "can_R"] & GL(R/I) \arrow[d, "can_{R/I}"] \arrow[dr] & \\
K_1(R) \arrow[r, "K_1(q)"'] & K_1(R/I) \arrow[r] & K_0(I)
\end{tikzcd}$$

we know $GL(q)(g) = \bar{g} \in E(R/I)$.

Since $E(R) \overset{E(q)}{\twoheadrightarrow} E(R/I)$ is surjective

$\exists\ e \in E(R)$ w/ $E(q)(e) = \bar{g}$.

So $GL(q)(ge^{-1}) = 1 \in E(R/I) \subset GL(R/I)$.

and $ge^{-1} \in GL(I)$.

Let $\mathrm{can}_I : GL(I) \twoheadrightarrow GL(I)/E(R,I)$

be the canonical surjection and

write

$$[ge^{-1}] = \mathrm{can}_I(ge^{-1}).$$

In sum, for every

$$g \in \ker(K_1(q))$$

there exists a $[ge^{-1}] \in K_1(I)$

such that $[ge^{-1}]$ maps to g,

so

$$\mathrm{im}\left(K_1(I) \to K_1(R)\right)$$

$$= \ker\left(K_1(R) \to K_1(R/I)\right).$$

$\square$

This exact sequence was known since the 1960's, but it wasn't known how to extend it to the left until Milnor defined K_2. It still didn't extend further until Quillen defined higher algebraic K-theory in 1972.