Lecture 12 : The Q-construction

I. The Q-construction

<u>Recall</u>. An exact category $\mathcal{E}$ is an additive subcategory $\mathcal{E} \subseteq \mathcal{A}$ of an abelian category $\mathcal{A}$ that is closed under extensions. We have a class of sequences

$$0 \to A \rightarrowtail C \twoheadrightarrow B \to 0$$

which are exact in $\mathcal{A}$ called admissible exact sequences and we say $A \rightarrowtail C$ is an admissible monomorphism and $C \twoheadrightarrow B$ is an admissible epimorphism.

<u>Remark</u>. In an exact category, the pullback

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, two heads] \\ A' \arrow[r, tail] & B' \end{tikzcd}$$

exists and it is also a pushout diagram up to isomorphism.

<u>Examples</u>

1) R ring, $P(R)$ = finitely generated projective R-modules
2) R ring, $M(R)$ = finitely generated R-modules
3) X scheme $VB(X)$ algebraic vector bundles over X
4) X scheme $M(X)$ coherent $\mathcal{O}_X$-modules.

Construction (Q-construction)

Given an exact category $\mathcal{C}$, we define a category $Q\mathcal{C}$ with the same objects and morphisms $A \to B$ in $Q\mathcal{C}$ given by ~~spans~~ isomorphism classes of spans

$$\begin{tikzcd}[column sep=small] & W \arrow[dl, two heads] \arrow[dr, tail] & \\ A & & B \end{tikzcd}$$

where two spans are in the same isomorphism class if the map of spans is of the form

$$\begin{tikzcd} A \arrow[d, equal] & W_1 \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\wr"] & B \arrow[d, equal] \\ A & W_2 \arrow[l, two heads] \arrow[r, tail] & B \end{tikzcd}.$$

Terminology/Notation: Given an admissible monomorphism $i : M \rightarrowtail M'$ we can form

$$i_! : \begin{tikzcd}[column sep=small] & M \arrow[dl, "\mathrm{id}_M"'] \arrow[dr] & \\ M & & M' \end{tikzcd}$$

and given an admissible epimorphism $j : M \twoheadrightarrow M'$ we form

$$j^! : \begin{tikzcd}[column sep=small] & M \arrow[dl] \arrow[dr, "\mathrm{id}_M"] & \\ M' & & M \end{tikzcd}$$

Write $i_M : 0 \rightarrowtail M$ in $\mathcal{C}$ and $j_M : M \twoheadrightarrow 0$ in $\mathcal{C}$.

Prop. There is a canonical isomorphism

$$\pi_1 BQ\mathcal{C} \cong K_0(\mathcal{C})$$

Pf. Recall that there is an equivalence of categories

$$\mathrm{Fun}(Q\mathcal{C},\mathrm{Set}) \supseteq \mathrm{Fun}'(Q\mathcal{C},\mathrm{Set}) \rightleftarrows \pi_1 Q\mathcal{C}\text{-sets}.$$

morphism inverting functors.

Let $\mathcal{F} \subseteq \mathrm{Fun}'(Q\mathcal{C},\mathrm{Set})$ denote the full subcategory on objects $F: Q\mathcal{C} \to \mathrm{Set}$ such that $F(M) = F(0)$ and $F(i_{M!}) = \mathrm{id}_M$.

<u>Step 1</u>. Show the inclusion $\mathcal{F} \rightleftarrows \mathrm{Fun}'(Q\mathcal{C},\mathrm{Set})$ is an equivalence of categories

<u>Step 2</u>. Show there is an equivalence of categories $\mathcal{F} \rightleftarrows K_0(\mathcal{C})\text{-set}$

Proof of 1). Given a morphism inverting functor F, we can form F' by letting $F'(M) = F(0)$ and $F(f) = \mathrm{id}_M$. Then the composite

$$\mathcal{F} \subseteq \mathrm{Fun}'(Q\mathcal{C},\mathrm{Set}) \longrightarrow \mathcal{F}$$ is the identity

up to isomorphism.

we define a natural transformation

$$\mathrm{Fun}'(Q\mathscr{C}, \mathrm{Set}) \times [1] \longrightarrow \mathrm{Fun}'(Q\mathscr{C}, \mathrm{Set})$$

from the other composite to the identity by

$$\begin{tikzcd}
F'(m) \arrow[r, "id_{F(0)}"] \arrow[d, "F(i_m!)^{-1}"'] \arrow[rr, bend left, "id_{F(0)}"] & F'(m') \arrow[r, "id_{F(0)}"] \arrow[d, "F(i_{m'}!)^{-1}"] & F'(0) \arrow[d, "id_{F(0)}"] \\
F(m) \arrow[r, "F(m\to m')"'] \arrow[rr, bend right, "F(i_m!)"'] & F(m') \arrow[r, "F(i_{m'})"'] & F(0)
\end{tikzcd}$$

<u>Proof of 2.</u>

Given a $K_0(\mathscr{C})$-set S, we define F_S by

$$F_S(m) = S, \quad F_S(i_m) = id_S,$$

$$F_S\left(m' \xleftarrow{p} m \rightarrowtail m\right) = S \xrightarrow{[\ker(p)]\cdot -} S$$

this defines a functor

$$\begin{aligned} K_0(\mathscr{C})\text{-Set} &\longrightarrow \mathscr{F} \\ S &\longmapsto F_S \end{aligned}.$$

Given a functor F in $\mathscr{F}$, and $i: m \rightarrowtail m'$ in $\mathscr{C}$

then $i \circ i_{m'} = i_m$ so
$$\begin{aligned} id_{F(0)} = F(i_m) &= F(i \circ i_{m'}) \\ &= F(i) \circ F(i_{m'}) \\ &= F(i) \circ id_{F(0)} \\ &= id_{F(0)}. \end{aligned}$$

Given an exact sequence

$$0 \longrightarrow m' \overset{i}{\rightarrowtail} m \overset{j}{\twoheadrightarrow} m'' \to 0$$

in $\mathscr{C}$

We have that $j^! i_{M''}^! = i_! j_{M'}^!$

$$\left[\begin{tikzcd} & & M' \arrow[dl] \arrow[dr, "i"] \arrow[dd, phantom, "\lrcorner" very near start] & & \\ & 0 \arrow[dl, equal] \arrow[dr, hook, "i_{M''}"'] & & M \arrow[dl, "j"] \arrow[dr, equal] & \\ 0 & & M'' & & M \end{tikzcd}\right] = \left[\begin{tikzcd} & & M' \arrow[dl, equal] \arrow[dr, equal] & & \\ & M' \arrow[dl, "j_{M'}"'] \arrow[dr, equal] & & M' \arrow[dl, equal] \arrow[dr, hook, "i"] & \\ 0 & & M' & & M \end{tikzcd}\right]$$

So

$$F(j^!) = F(j^! i_{M''}^!) = F(i_! j_{M'}^!) = F(j_{M'}^!)$$

in $\mathrm{Aut}(F(0))$. Also, $j_M^! = j^! j_{M''}^!$

$$\left[\begin{tikzcd} & & M \arrow[dl, equal] \arrow[dr, equal] & & \\ & M \arrow[dl] \arrow[dr, equal] & & M \arrow[dl, equal] \arrow[dr, equal] & \\ 0 & & M & & M \end{tikzcd}\right] = \left[\begin{tikzcd} & & M \arrow[dl] \arrow[dr, equal] & & \\ & M'' \arrow[dl] \arrow[dr, equal] & & M \arrow[dl] \arrow[dr, equal] & \\ 0 & & M'' & & M \end{tikzcd}\right]$$

So

$$\begin{aligned} F(j_M^!) &= F(j^! \circ j_{M''}^!) \\ &= F(j^!) \circ F(j_{M''}^!) \\ &= F(j_{M'}^!) \circ F(j_{M''}^!) \end{aligned}$$

by the universal property of $K_0(\mathscr{C})$ there is a group homomorphism

$$\begin{aligned} K_0(\mathscr{C}) &\longrightarrow \mathrm{Aut}(F(0)) \\ [M] &\longmapsto F(j_M^!) \end{aligned}$$

Recall: Suppose φ in the diagram

$$\begin{tikzcd} \mathrm{ob}\,\mathcal{C} \arrow[r] \arrow[dr, bend right, "\varphi"'] & K_0(\mathcal{C}) \arrow[d, dotted, "\exists!\, f"] \\ & A \in Gp \quad (\mathrm{im}\,\varphi \subset Ab) \end{tikzcd}$$

satisfies forall admissible exact sequences

$$0 \to M' \rightarrowtail M \twoheadrightarrow M'' \to 0$$

then $\varphi(M) = \varphi(M') \cdot \varphi(M'')$

then there exists a unique factorization f.

There fore, there are funltors

$$K_0(\mathcal{C})\text{-Set} \rightleftarrows \mathcal{F}$$

$$S \longmapsto F_S$$

$$F(0) \longleftarrow\!\shortmid F$$

and we can easily check that this gives an equivalence of categories.

Def. $K^Q(\mathcal{C}) := \Omega BQ\mathcal{C}$

Thm: ($+=Q$) when $\mathcal{C} = P(R)$, there is a homotopy equivalence

$$K_0(R) \times BGL(R)^+ \simeq K^Q(P(R)).$$

Remark: This homotopy equivalence passes through another construction, the $S^{-1}S$ construction:

$$K_0(R) \times BGL(R)^+ \simeq B(\mathrm{iso}P(R))^{-1}(\mathrm{iso}P(R)) \simeq K^Q(P(R))$$

and one can prove these are equivalences by proving that the right two satisfy the universal property of $BGL(R)^+$ on each path component.

<u>Thm 1.</u> Let $\mathcal{C}$ be an exact category regarded as a Waldhausen category $(\mathcal{C}, c\mathcal{C}, iso\mathcal{C})$ with cofibrations the admissible monomorphisms and weak equivalences isomorphisms. Then there is a homotopy equivalence

$$\begin{array}{ccc} K^{w}(\mathcal{C}) & \simeq & K^{Q}(\mathcal{C}) \\ \| & & \| \\ \Omega\,|N_\cdot wS_\cdot\mathcal{C}| & & \Omega B Q\mathcal{C} \end{array}$$

To prove this, we will first introduce a functor called the edgewise subdivision

$$\begin{aligned} \mathcal{C}^{\Delta^{op}} &\longrightarrow \mathcal{C}^{\Delta^{op}} \\ X_\cdot &\longmapsto X_\cdot^{e} \end{aligned}$$

<u>Def.</u> We define a functor

$$sd^e : \Delta \to \Delta \quad \text{by}$$

$$sd^e([k]) = [2k+1]$$
$$sd^e(\alpha : [n] \to [m]) : [2n+1] \to [2m+1]$$
$$sd^e(\alpha)(s) = \begin{cases} \alpha(s) & 0 \leq s \leq n \\ \alpha(s-(n+1)) + m+1 & n+1 \leq s \leq m \end{cases}$$

Given a simplicial object

$$X_\bullet : \Delta^{op} \to \mathcal{C}$$

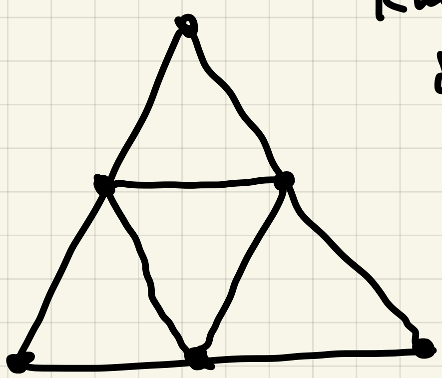
in $\mathcal{C}$, we write

$$X_\bullet^e : \Delta^{op} \xrightarrow{(sd^e)^{op}} \Delta^{op} \xrightarrow{X_\bullet} \mathcal{C}$$

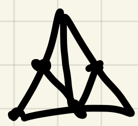
for the edgewise subdivision of $X_\bullet$.

Ex: $X_\bullet = \Delta^2$

$$(\Delta^2)^e =$$



Rmk: This is different from Segal's subdivision



We write d_i^e and s_i^e for the face and degeneracies of $X_\bullet^e$. These satisfy the following compatibility

$$\begin{tikzcd} X_k^e \arrow[r, "d_i^e"] \arrow[d, equal] & X_{k-1}^e \arrow[d, equal] \\ X_{2k+1} \arrow[r, "d_{k-i} \circ d_{k+i+1}"] & X_{2k-1} \end{tikzcd}$$

$$\begin{tikzcd} X_k^e \arrow[r, "s_i^e"] \arrow[d, equal] & X_{k+1}^e \arrow[d, equal] \\ X_{2k+1} \arrow[r, "s_{k-i} \circ s_{k+i+1}"] & X_{2k+1} \end{tikzcd}$$

<u>Thm</u> 2. Let X be a simplicial set. There is a canonical homeomorphism

$$|X_\bullet| \cong |X_\bullet^e| .$$

<u>Pf sketch</u>: Check this for $X_\cdot = \Delta^1$ by explicit calculation. Show Δ^p is a retract of $\prod_p \Delta^1$ to prove the result for Δ^p. Then use the fact that

$$|sd^e X_\cdot| \cong |sd^e(\operatorname{colim}_{p \in \Delta^{op}/X_\cdot} \Delta^p)| \cong |\operatorname{colim}_{p \in \Delta^p/X_\cdot} sd^e(\Delta^p)| \cong |\operatorname{colim}_{p \in \Delta^p/X_\cdot} \Delta^p| \cong |X_\cdot|.$$

<u>Notation</u>:

Write $\mathrm{iso}\, N_\cdot Q\mathscr{C}$ for the simplicial category $\mathrm{Fun}([n], \mathrm{iso}\, Q\mathscr{C})$

<u>Lemma 1</u>. There is a homotopy equivalence

$$BQ\mathscr{C} \xrightarrow{\simeq} |N_\cdot \mathrm{iso}\, Q\mathscr{C}|$$

<u>Pf</u>. This is left as an exercise since it is proven in a very similar way to the result

$$|s_\cdot \mathscr{C}| \xrightarrow{\simeq} |\mathrm{iso}\, S_\cdot \mathscr{C}|$$

that we discussed earlier. $\square$

Lemma 2. There is a map of simplicial categories

$$w_k : \mathrm{iso}\, S_{2k+1}\mathcal{C} \longrightarrow \mathrm{iso}\, N_k Q\mathcal{C}$$

which is an equivalence of categories for each $k \geq 0$.

Proof of theorem 1.

The homotopy equivalence

$$|N_\bullet \mathrm{iso}\, S_\bullet \mathcal{C}| \overset{\text{Thm 2}}{\simeq} |N_\bullet (\mathrm{iso}\, S_\bullet \mathcal{C})^e| \xrightarrow[\simeq]{\text{Lemma 2}} |N_\bullet \mathrm{iso}\, Q\mathcal{C}| \xrightarrow[\simeq]{\text{Lemma 1}} BQ\mathcal{C}$$

induces a homotopy equivalence

$$K^{\omega}(\mathcal{C}) := \Omega|N_\bullet \mathrm{iso}\, S_\bullet \mathcal{C}| \xrightarrow{\simeq} \Omega BQ\mathcal{C} =: K^{Q}(\mathcal{C}).$$

We break Lemma 2 into several parts. First, we define the functor

$$\mathrm{iso}\, S_{2k+1}\mathcal{C} \longrightarrow \mathrm{iso}\, Q_k\mathcal{C}$$

on objects and leave it as an exercise to check that it is defined on morphisms. We send $A \in S_{2k+1}\mathcal{C}$

$$A = \left(\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{0,k+1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{0,2k+1} \arrow[d, two heads] \\
& A_{12} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1,k+1} \arrow[r, tail] \arrow[d, two heads] & \cdots & \vdots \\
& & \ddots & \vdots \arrow[d, two heads] & & \vdots \\
& & & A_{k,k+1} \arrow[r, tail] & \cdots & \vdots \arrow[d, two heads] \\
& & & & \ddots & A_{2k,2k+1}
\end{tikzcd}\right)$$

to the span

$$\begin{tikzcd}
& A_{k-1,k+1} \arrow[dl, two heads] \arrow[dr, tail] & & A_{k-2,k+2} \arrow[dl, two heads] \arrow[dr, tail] & & \cdots \arrow[dl, two heads] \arrow[dr, tail] & \\
A_{k,k+1} & & A_{k-1,k+2} & & \cdots & & A_{0,2k+1}
\end{tikzcd}$$

sitting inside $S_{2k+1}\mathcal{C}$.

<u>Lemma 3</u>. The map

$$W_k : \mathrm{iso}\, S_{2k+1}\mathcal{C} \longrightarrow \mathrm{iso}\, N_k Q\mathcal{C}$$

is a map of simplical categories.

<u>Pf.</u> I will leave this to you to check, but I will give an example.

$\in S_3\mathcal{C}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\in N_1 Q\mathcal{C}$

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] & A_{03} \arrow[d, two heads] \\ & A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\ & & A_{23} \end{tikzcd} \longmapsto \begin{tikzcd} A_{02} \arrow[r, tail] \arrow[d, two heads] & A_{03} \\ A_{12} \end{tikzcd}$$

$\downarrow d_2$

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[d, two heads] \\ & A_{12} \end{tikzcd} \qquad d_0^e = d_0 \circ d_2 \qquad\qquad d_0$$

$\downarrow d_0$

$$A_{12} \longmapsto A_{12}$$

We first observe that given a span,

$$\begin{tikzcd}[column sep=small]
 & A_{k-1,k+1} \arrow[dl, two heads] \arrow[dr, tail] & & A_{k-2,k+2} \arrow[dl, two heads] \arrow[dr, tail] & & \cdots \arrow[dl, two heads] \arrow[dr, tail] & \\
A_{k,k+1} & & A_{k-1,k+2} & & \cdots & & A_{0,2k+1}
\end{tikzcd}$$

we can reconstruct all of A by taking pushouts and pullbacks.

Ex:

$$\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] & A_{03} \arrow[d, two heads] & & A_{02} \arrow[r, tail] \arrow[d, two heads] & A_{03} \\
 & A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] & \longmapsto & A_{12} & \\
 & & A_{23} & & &
\end{tikzcd}$$

$$\begin{tikzcd}
A_{01} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near end] & A_{03} \arrow[d] \\
0 \arrow[r] & A_{12} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & A_{13} \arrow[d] \\
 & 0 \arrow[r] & A_{23}
\end{tikzcd}$$

So since pullbacks of the form

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads] \\
X' \arrow[r, tail] & Y'
\end{tikzcd}$$

are also pushouts in $\mathcal{C}$ up to isomorphism, we have shown:

Lemma 4.

The map

$$\mathrm{ob}(\mathrm{iso}\, S_{2k+1}\mathscr{C}) \longrightarrow \mathrm{ob}(\mathrm{iso}\, Q_k\mathscr{C})$$

is surjective.

To prove Lemma 2. it suffices to show the following:

Lemma 5.

The functor

$$W_k : \mathrm{iso}\, S_{2k+1}\mathscr{C} \longrightarrow \mathrm{iso}\, N_k Q\mathscr{C}$$

is fully faithful.

Proof. We need to show that W_k induces a bijection

$$\mathrm{iso}\, S_{2k+1}\mathscr{C}(A, A') \xrightarrow{\cong} \mathrm{iso}\, N_k Q\mathscr{C}(W_k(A), W_k(A')).$$

Step 1. (Surjectivity)

Given a morphism

$$
\begin{tikzcd}
& A_{k-1,k+1} \arrow[dl, two heads] \arrow[dr, tail] \arrow[ddd, "\cong"] & & A_{k-2,k+3} \arrow[dl, two heads] \arrow[dr, tail] \arrow[ddd, "\cong"] & & \cdots \arrow[dl, two heads] \arrow[dr, tail] & \\
A_{k,k+1} \arrow[d, "\cong"] & & A_{k-1,k+2} \arrow[d, "\cong"] & & \cdots & & A_{0,k+1} \arrow[d, "\cong"] \\
A'_{k,k+1} & & A'_{k-1,k+2} & & \cdots & & A'_{k,k+1} \\
& A'_{k-1,k+1} \arrow[ul, two heads] \arrow[ur, tail] & & A'_{k-2,k+3} \arrow[ul, two heads] \arrow[ur, tail] & & \cdots \arrow[ul, two heads] \arrow[ur, tail] &
\end{tikzcd}
$$

in $\mathrm{iso}\, Q_k \mathscr{C}$ we can do the same inductive procedure of taking pullbacks and pushouts to define a map

$$A \longrightarrow A' \quad \text{in } \mathrm{iso}\, S_{2k+1}\mathscr{C}$$

since pullbacks and pushouts preserve isomorphisms.

Step 2. (Injectivity)

Suppose

$$t_0, t_1 : A \rightarrow A'$$

are maps in iso $S_{2k+1}\mathcal{C}$ such that

$$W_k(t_0) = W_k(t_1).$$

Then we know

$$(t_0)_{i,j} = (t_1)_{ij} : A_{ij} \rightarrow A'_{ij}$$

when $i+j = 2k$ and $i+j = 2k+1$.

Since pullbacks and pushouts are functorial they also preserve identities so we can do the same iterative procedure to reconstruct

$$t_0, t_1 : A_{ij} \rightarrow A'_{ij}$$

and show that $t_0 = t_1$.

Examples.

(1) $P(R)$ we write $K(R) := K^Q(P(R))$

(2) $M(R)$ we write

$$G(R) := K^Q(M(R))$$

(3) $VB(X)$ X scheme

$$K(X) := K^Q(VB(X))$$

(4) $M(X)$ X Noetherian scheme

$$G(X) := K^Q(M(X))$$

(5) $Ch^b(P(R))$

$$K(R) \simeq K(Ch^b(P(R)))$$

(6) $Ch^b(M(R))$

$$G(R) \simeq K(Ch^b(M(R)))$$

Gillet-Waldhausen theorem.

Coming soon. We will prove

Reduction by resolution.

Cor. R a Noetherian regular ring then there is a homotopy equivalence

$$K(R) \xrightarrow{\simeq} G(R).$$

Dévissage.

Cor. Let R be an artinian local ring with maximal ideal m (so that $m^r = 0$ for some $r \geq 1$) and quotient field $R/m = k$

Ex: $\mathbb{Z}/p^n$

Then

$$G(R) \simeq K(k).$$

Localization.

Cor. Let R be a Dedekind domain with fraction field F and residue fields R/p. Then there is a LES

$$\to \bigoplus_p K_i(R/p) \to K_i(R) \to K_i(F) \to \bigoplus_p K_{i-1}(R/p) \to K_{i-1}(R) \to K_{i-1}(F) \to \cdots$$

Let $\mathbb{F}_q$ be a finite field, then we will show that

$$K_i(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/q^k - 1 & i = 2k-1 \quad k \geq 1 \\ 0 & \text{o.w.} \end{cases}$$