Lecture 3 :

Milnor K-theory

I. K_2 of a ring

Def: Let A be a ring, then let $St_n(A)$ be the free group on generators $x_{ij}(a)$ for $a \in A$, $1 \leq i \neq j \leq n$ modulo relations

(1) $x_{ij}(a) \cdot x_{ij}(b) = x_{ij}(a+b)$

(2) $[x_{ij}(a), x_{kl}] = \begin{cases} 1 & \text{if } j \neq k,\ i \neq l \\ x_{il}(rs) & \text{if } j = k \ \ i \neq l \\ x_{kj}(-sr) & \text{if } j \neq k \ \ i = l \end{cases}$

called the

Steinberg relations.

Exercise: Show $e_{ij}(a) \in E_n(A)$ satisfy the Steinberg relations for $n \geq 3$.

Consequently, there is a canonical surjection

$$St_n(A) \to E_n(A).$$

Note: The Steinberg relations for $k < n$ are contained in the relations for n, so there are group homorphisms

$$St_{n-1}(A) \to St_n(A).$$

These group homomorphisms are compatible with the canonical surjections

$$\begin{tikzcd} St_n(A) \arrow[r, two heads] \arrow[d] & E_n(A) \arrow[d] \\ St_{n+1}(A) \arrow[r, two heads] & E_{n+1}(A) \end{tikzcd} \qquad n \geq 3$$

Exercise: Given compatible surjective maps

$$\begin{tikzcd} B_i \arrow[r, two heads] \arrow[d] & C_i \arrow[d] \\ B_{i+1} \arrow[r, two heads] & C_{i+1} \end{tikzcd}$$

of groups $\forall\, i >> 0$, then the induced map $\operatorname{colim}_i B_i \twoheadrightarrow \operatorname{colim}_i C_i$ is a surjection.

Def:

$$K_2(A) := \ker(St(A) \to E(A)).$$

Note: By construction, there is an exact sequence

$$1 \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to 1.$$

Thm: [Steinberg]

The group $K_2(A)$ is exactly the center of $St(A)$ and consequently it is an abelian group.

We will prove a generalization of the following result later in the course.

<u>Thm</u> Let A be a Dedekind domain w/ field of fractions F, then there is an exact sequence

$$K_2(F) \to \prod_{p \in \mathbb{P}} K_1(A/p) \to K_1(A) \to K_1(F) \to \prod_{p \in \mathbb{P}} K_0(A/p) \to K_0(A) \to K_0(F) \to 0$$

where $\mathbb{P} = \{ p \subseteq A \mid p \text{ prime ideal} \}$.

II Milnor K-theory

<u>Construction</u>: Given an abelian group M, we define the <u>tensor algebra</u> of M

$$T(M) := \bigoplus_{i \geq 0} M^{\otimes i}$$

w/ underlying abelian group $\bigoplus_{i \geq 0} M^{\otimes i}$ and multiplication

$$T(M) \otimes T(M) \longrightarrow T(M) = \bigoplus_{k \geq 0} M^{\otimes k}$$

$$\Big(\bigoplus_{i \geq 0} M^{\otimes i}\Big) \otimes \Big(\bigoplus_{j \geq 0} M^{\otimes j}\Big)$$

$$\bigoplus_{n \geq 0} \bigoplus_{i+j=n} M^{\otimes i} \otimes M^{\otimes j}$$

induced by $M^{\otimes i} \otimes M^{\otimes j} \xrightarrow{\cong} M^{\otimes i+j}$.

(Note: $M^{\otimes 0} = \mathbb{Z}$.)

We grade $T(M)$ by letting elts

$$x \in \bigoplus_{i \geq 0} M^{\otimes i}$$

have grading degree

$$|x| = n \quad \text{if} \quad x \in M^{\otimes n} \hookrightarrow \underset{\substack{\parallel \\ T(M)}}{\bigoplus_{i \geq 0} M^{\otimes i}}$$

So $T(M)$ is a graded ring.

Ex: Let k be a field.

Then $k^\times$ is an abelian group and we can consider

$$T(k^\times).$$

Notation:

when $x \in k^\times$, write

$$\ell(x) \otimes \ell(x') \in k^\times \otimes k^\times \subseteq T(k^\times)$$

(just to distinguish it from an elt: $x \in k^\times \in T(k^\times)$ in degree 1)

<u>Def</u>: We define the Milnor K-theory of a field k by

$$K^M_*(k) = T(k^\times) \Big/ \left(\ell(x) \otimes \ell(1-x) : 1 \neq x \in k^\times \right)$$

Note:

$$K_0^M(k) = \mathbb{Z} = K_0(k)$$

$$K_1^M(k) = k^\times = K_1(k)$$

Exercise.

<u>Thm</u> [Matsumoto]

For any field k,

$$K_2^M(k) = K_2(k).$$

(Note that Matsumoto's theorem came first and inspired Milnor's definition of K_2 and higher K-groups.)

<u>Proposition</u>: The Milnor K-groups of a finite field $\mathbb{F}_q$ are

$$K^M_*(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{F}_q^\times$$

trivial square zero extension where $\mathbb{F}_q^\times$ is in degree 1.

in particular

$K^M_k(\mathbb{F}_q) = 0$ for $k \geq 2$.

<u>Proof</u>: First, we will show that

$$\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times \Big/ \left(x \otimes (1-x) \; : \; \substack{x \in \mathbb{F}_q^\times \\ x \neq 1} \right) = 1.$$

$$\parallel \leftarrow \text{Matsumoto}$$

$$Z(St(\mathbb{F}_q))$$

Write $\cdot$ for the group operation in $Z(St(\mathbb{F}_q))$ and e for the unit (corresponding to

$$1\otimes 1 \in \mathbb{F}_q^\times \otimes \mathbb{F}_q^\times \Big/ \left(x\otimes(1-x) \ \ \begin{matrix} x\in\mathbb{F}_q^\times, \\ x\neq 1 \end{matrix} \right)$$

Note: $\mathbb{F}_q^\times \cong \mathbb{Z}/q-1$

and

$$\mathbb{F}_q^\times \underset{\mathbb{Z}}{\otimes} \mathbb{F}_q^\times \cong \mathbb{Z}/q-1 \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/q-1 \cong \mathbb{Z}/q-1$$

$$x\otimes x \longmapsto g\otimes g \longmapsto g$$

x generates $\mathbb{F}_q^\times$ $\qquad$ g generates $\mathbb{Z}/q-1$

So $x\otimes x$ generates

$$\mathbb{F}_q^\times \underset{\mathbb{Z}}{\otimes} \mathbb{F}_q^\times .$$

It suffices to show that

$$[x\otimes x] = [1\otimes 1] \in \mathbb{F}_q^\times \otimes \mathbb{F}_q^\times \Big/ \left(x\otimes(x-1) ; \ x\neq 1 \in \mathbb{F}_q^\times \right).$$

Case 1: If q is even,

$2x = 0$ in $\mathbb{F}_q$ so

$$x \otimes x = x \otimes -x$$

and consequenly

$$[x \otimes x] = [x \otimes -x]$$

$$= [x \otimes 1] \quad (g \otimes 0 = 0 \in \mathbb{Z}/_{q-1} \otimes \mathbb{Z}/_{q-1})$$

$$= e \in K_2(\mathbb{F}_q).$$

More generally,

$$(\mathbb{F}_q^\times)^{\otimes n} \cong \mathbb{Z}/q-1$$

w/ generator $\underbrace{x \otimes \ldots \otimes x}_{n}$ and

$$[\underbrace{x \otimes x \otimes \ldots \otimes x}_{n}] = [\underbrace{x \otimes \ldots \otimes -x}_{n}]$$

$$= [x \otimes \ldots \otimes 1]$$

$$= e \in K_n(\mathbb{F}_q)$$

$n \geq 1$.

Case 2:

Observe that
$$[y \otimes -y] = [y \otimes 1] = e$$
$$[x \otimes -x] = [x \otimes 1] = e$$

implies
$$[x \otimes y] \cdot [y \otimes x] = [x \otimes -xy] \cdot [y \otimes -xy] = [xy \otimes -xy] = e$$

In particular, $[x \otimes x]^2 = e$

More generally,
$$[x \otimes x]^{mn} = [x^m \otimes x^n]$$

m, n odd.

Given a non-square $u \in \mathbb{F}_q - \{0,1\}$

such that $1-u$ is also a non-square

in $\mathbb{F}_q - \{0,1\}$,

then nontrivial elt (if one exists) in $K_2(\mathbb{F}_q)$ can

be written as

$$[u \otimes 1-u] = [x \otimes x]^{nm} \cdot [x \otimes x]^{j}$$

$$\begin{array}{cc} \shortparallel & \shortparallel \\ x^n & x^m \end{array} = [x \otimes x]^{nm+j}$$

But then, these are also trivial

because $[u \otimes 1-u] = e$.

We therefore just need to

show

$\exists\ u \in \mathbb{F}_q - \{0,1\}$ a nonsquare

such that $1-u$ is also a nonsquare

The assignment

$$u \longmapsto 1-u$$

defines a C_2-action

$$\mathbb{F}_q - \{0,1\} \xrightarrow{u \longmapsto 1-u} \mathbb{F}_q - \{0,1\}.$$

$$\# \mathbb{F}_q - \{0,1\} = q-2$$

and there are $(q-1)/2$ nonsquares,

but only $(q-3)/2 = (q-1)/2 - 1$

squares. So $\exists$ such a u.

<u>Def</u>: The Brauer group of a field k denoted $Br(k)$

is generated by isomorphism

classes of central simple algebras

modulo

1) $[A \otimes_F B] = [A] \cdot [B]$

2) $[M_n(A)] = 0$

(See K-book p. 57 – 59
for more details.)

<u>Prop</u>: If K contains a primitive n-th root of unity, there is a group homomorphism

$$K_2(K) \to Br(K)$$
$$[\alpha \otimes \beta] \mapsto [A_\zeta(\alpha, \beta)]$$

where $A_\zeta = K\langle x, y\rangle \Big/ \left(\begin{array}{c} x^n = \alpha\cdot 1,\ y^n = \beta\cdot 1 \\ yx = \zeta x y \end{array} \right)$

Since $[A_\zeta^{\otimes n}] = [M_j(K)] = 0 \in Br(K)$
(Thm 8.12 Jacobson "Basic Algebra II")

this factors as

$$K_2(K) \big/ nK_2(K) \longrightarrow Br(K)$$

called the "power norm residue symbol".

By Merkurjev-Suslin,

$$K_2(k)/_{n}K_2F \cong {}_n\mathrm{Br}(k)$$

n-torsion in $\mathrm{Br}(F)$

Note that $\mathbb{F}_q$ contains a primitive n-th root of unity $\forall n \geq 1$ such that $n \mid q-1$.

<u>Cor</u>:

$${}_n\mathrm{Br}(\mathbb{F}_q) = 0$$

for all $n \mid q-1$