Lecture 13: Resolution and Dévissage

I. Resolution

Recall: R (right Noetherian) ring, then

$$K(R) := K^a(\mathcal{P}(R)) \quad \text{and} \quad G(R) := K^a(\mathcal{M}(R)).$$

Q. When is $K(R) \xrightarrow{\sim} G(R)$?

<u>Def</u>. We say R is regular if every finitely generated R-module has a <u>finite</u> resolution by finitely generated projective R-modules.

A. When R is a regular ring, then $K(R) \xrightarrow{\sim} G(R)$.

<u>General setup</u>. Let $\mathcal{M}$ be an exact category and let $\mathcal{P}$ be a sub additive category such that whenever

$$0 \to M'' \rightarrowtail M \twoheadrightarrow M' \to 0$$

is exact in $\mathcal{M}$ and $M'', M' \in \mathcal{P}$ then $M \in \mathcal{P}$.

This gives an exact category structure on $\mathcal{P}$ and the inclusion is an exact functor.

We will say $(\mathcal{P}, \mathcal{M})$ is pre-resolvable if whenever $0 \to M'' \rightarrowtail M \twoheadrightarrow M' \to 0$ is exact in $\mathcal{M}$ and $M \in \mathcal{P}$ then $M'' \in \mathcal{P}$.

We will say $(\mathcal{P}, \mathcal{M})$ is resolvable if, in addition, for every $M \in \mathcal{M}$ there is an admissible sequence
$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$
in $\mathcal{M}$ with $P_i \in \mathcal{P}$ for $0 \le i \le n$.

<u>Thm</u> (Resolution)
When $(\mathcal{P}, \mathcal{M})$ is resolvable, the inclusion induces a homotopy equivalence
$$K(\mathcal{P}) \xrightarrow{\simeq} K(\mathcal{M}).$$

Ex:

(1) Let R be a regular ring, then $(\mathcal{P}(R), \mathcal{M}(R))$ is resolvable and
$$K(R) \xrightarrow{\simeq} G(R).$$

(2) Let X be a separated regular Noetherian scheme, then
$$K(X) \xrightarrow{\simeq} G(X).$$

First, we will prove a special case.

<u>Prop</u>. 1. Suppose $(\mathcal{P}, \mathcal{M})$ is pre-resolvable and for every $M \in \mathcal{M}$ there is a short exact sequence

$$0 \to P_1 \rightarrowtail P_0 \to M \to 0$$

such that P_0 (and consequently P_1) are isomorphic to objects in $\mathcal{P}$. Then

$$K(\mathcal{P}) \xrightarrow{\simeq} K(\mathcal{M}).$$

<u>Proof of the resolution theorem.</u>

Let $\mathcal{M}_i \subseteq \mathcal{M}$ be the full subcategory of objects $M \in \mathcal{M}$ s.t. there exists a resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

with $n \leq i$ and P_j isomorphic to an object in $\mathcal{P}$ for all $0 \leq j < n$.

Lemma 1. For each $n \geq 0$, the following hold for an exact sequence

$$0 \to M'' \rightarrowtail M \twoheadrightarrow M' \to 0$$

in $\mathcal{M}$:

(1) $M \in \mathcal{M}_n$, $M' \in \mathcal{M}_{n+1} \Rightarrow M'' \in \mathcal{M}_n$

(2) $M'', M' \in \mathcal{M}_{n+1} \Rightarrow M \in \mathcal{M}_{n+1}$

(3) $M, M' \in \mathcal{M}_n \Rightarrow M'' \in \mathcal{M}_n$

Proof. This is left as an exercise.

Assuming the lemma then, since we know $\exists$ an admissible epimorphism $P \twoheadrightarrow M$, whenever $M \in \mathcal{M}_{n+1}$ then $\ker(P \twoheadrightarrow M) \in \mathcal{M}_n$ by (1). By (2) and (3) we know $(\mathcal{M}_{n+1}, \mathcal{M}_n)$ is pre-resolvable and (1) then says that $(\mathcal{M}_{n+1}, \mathcal{M}_n)$ satisfies the hypotheses of Prop. 1. Consequently,

$$K(P) \xrightarrow{\simeq} K(\mathcal{M}_1) \to \ldots \to \operatorname{colim}_i K(\mathcal{M}_i) \xrightarrow{\simeq} K(\mathcal{M}).$$

It therefore suffices to prove the proposition.

<u>Proof of Prop. 1</u>. Note that $QP \subseteq QM$ is <u>not</u> the inclusion of a full subcategory since if $p, p' \in P$ then there exist

$$p \overset{id_p}{\twoheadleftarrow} p \rightarrowtail p'$$

in QM where $\operatorname{coker}(A \rightarrowtail B) \notin P$ and consequently this is not a map in QP.

Let Q be the full subcategory of QM on objects of QP so that we have inclusions

$$QP \subseteq Q \subseteq QM.$$

Then it suffices to show that

$$BQP \simeq BQ \simeq BQM$$

Let $i : QP \hookrightarrow Q$ and $j : Q \hookrightarrow QM$ be the canonical inclusions.

<u>Step 1.</u> $BQP \xrightarrow[Bi]{\simeq} BQ$

It suffices, by Quillen's Theorem A, to show that

$$Bi/p \simeq *$$

for each P in Q; i.e each P in P.

An object in i/p is a pair

$$(P_2, \, u: P_2 \twoheadleftarrow P_1 \rightarrowtail P)$$

where P/P_1 is in M. Define a functor

$$z: i/p \to i/p$$

by $z(P_2, u) = (P_1, \, P_1 = P_1 \rightarrowtail P)$.

Then there are natural transformations

$$id_{i/p} \Rightarrow z \Leftarrow Const_{(0, \, 0 = 0 \rightarrowtail P)}$$

given by

$$(P_2, P_2 \twoheadleftarrow P_1 \rightarrowtail P) \to (P_1, P_1 = P_1 \rightarrowtail P) \leftarrow (0, \, 0 = 0 \rightarrowtail P)$$

Thus, $id_{Bi/p} \simeq Const_{(0, 0=0 \rightarrowtail P)}$ so $Bi/p \simeq *$ for all $P \in P$.

<u>Step 2</u>: We now apply the dual of Quillen's theorem A. We need to show that $BM\backslash j \simeq 0$ for all M in $\mathcal{M}$.

We first make a reduction. Let

$\mathcal{B} \subseteq M\backslash j$ be the full subcategory

on objects $(P, u: M \twoheadleftarrow P_1 \rightarrowtail P)$ with $P \in \mathcal{P}$ ($\Rightarrow P_1 \in \mathcal{P}$) (since there exists an epimorphism $M \twoheadleftarrow P_1$ for some P_1 in $\mathcal{P}$ this is nonempty; i.e. $(P_1, M \twoheadleftarrow P_1 = P_1) \in \mathcal{B}$.

There is a right adjoint

$$r(P, u) = (P_1, M \twoheadleftarrow P_1 = P_1).$$

Thus, $B\mathcal{B} \simeq B\, M\backslash j$. It suffices to show $B\mathcal{B} \simeq *$ (for any choice of M).

Fix $(P_0, u_0: M \twoheadleftarrow P_0 = P_0)$ an admissible epimorphism $P_0 \twoheadrightarrow M$, which we can do by assumption.

we define a functor

$$p : \mathscr{C} \to \mathscr{C}$$

by $p(P, u : M \twoheadleftarrow P = p)$

$$\begin{array}{c} \| \\ (P \times_M P_0 , M \twoheadleftarrow P \times_M P_0 = P \times_M P_0) \end{array}$$

(Note: since $\mathcal{P}$ is closed under subobjects $P_1 \in \mathcal{P}$

$$\begin{tikzcd}
P_1 \arrow[r, tail] \arrow[d, equal] & P \times_M P_0 \arrow[r] \arrow[d] & P \arrow[d, two heads] \\
P_1 \arrow[r, tail] & P_0 \arrow[r, two heads] & M
\end{tikzcd}$$

and since $\mathcal{P}$ is closed under extensions

$P \times_M P_0$ is in $\mathcal{P}$.)

we then define natural transformations

$$\mathrm{id}_{\mathscr{C}} \Leftarrow p(-) \Rightarrow \mathrm{const}_{(P_0, u_0)} \quad \text{by}$$

$$\begin{array}{c} (P, v : M \twoheadleftarrow P = p) \leftarrow (P \times_M P_0 , M \twoheadleftarrow P \times_M P_0 = P \times_M P_0) \\ \downarrow \\ (P_0 , u_0) \end{array}$$

.

Therefore,

$$\mathrm{id}_{B\mathscr{C}} \simeq \mathrm{const}_{(P_0, \varphi_0)}$$

so $B\mathscr{C} \simeq *$ for any choice of M. $\square$

<u>Def.</u> Let X be a separated Noetherian scheme. We say X is <u>regular</u> if algebraic vector bundle over X $\mathcal{F}$ has a finite resolution by coherent $\mathcal{O}_X$-modules.

<u>Recall.</u>

$$K(X) := K^{\mathcal{Q}}(VB(X))$$

$$G(X) := K^{\mathcal{Q}}(M(X))$$

<u>Cor.</u> When X is a separated regular Noetherian scheme

$$K(X) \simeq G(X).$$

Cor: When R is a regular left Noetherian ring, then
$$K(R) \simeq G(R).$$

II. Dévissage

Let $\mathcal{A} \subseteq \mathcal{B}$ be a sub abelian category which is the inclusion of a full subcategory. We say $\mathcal{A}$ is an exact abelian subcategory if the inclusion $i: \mathcal{A} \subseteq \mathcal{B}$ is an ab enriched functor $\left(\mathrm{Hom}_{\mathcal{A}}(A, A') < \mathrm{Hom}_{\mathcal{B}}(i(A), i(A'))\right)$ and it preserves exact sequences. We will say that $\mathcal{A} \subseteq \mathcal{B}$ is pre-unscrewable if, in addition, $\mathcal{A}$ is closed under subobjects and quotients.

We say (A, B) as above is unscrewable if, in addition, every object B has a finite filtration

$$
\begin{array}{ccccccccccc}
0 = B_n & \subseteq & B_{n-1} & \subseteq & \cdots & \subseteq & B_1 & \subseteq & B_0 & = & B \\
 & & \wr\wr & & & & \downarrow & & \downarrow & & \\
 & . & B_{n-1}/B_n & & & & B_1/B_2 & & B_0/B_1 & &
\end{array}
$$

where each of B_i/B_{i+1} is in A.

<u>Thm</u> [Dévissage)

Suppose (A, B) is unscrewable, then

$$K(A) \xrightarrow{\sim} K(B).$$

First, we will discuss some applications.

Examples.

1) If I is a nilpotent ideal in a Noetherian ring R, then $G(R/I) \simeq G(R)$.

2) Let $\mathcal{T}$ be the abelian category of finitely generated torsion modules over a Dedekind domain R. Let T_{ss} be the exact abelian subcategory of semisimple objects in $\mathcal{T}$. Then (T_{ss}, T) is unscrewable so

$$K(T_{ss}) \simeq K(T).$$

Moreover, by Shur's lemma there is an exact equivalence of categories

$$T_{s.s} \rightleftarrows \prod_{p} M(R/p).$$

So
$$
\begin{aligned}
K^{\mathbb{Q}}(T) &\simeq K^{\mathbb{Q}}(T_{ss}) \\
&\simeq K^{\mathbb{Q}}\Big(\prod_p M(R/p)\Big) \\
&\simeq \prod_p K^{\mathbb{Q}}(M(R/p)) \\
&= \prod_p G(R/p) \\
&\simeq \prod_p K(R/p)
\end{aligned}
$$
by the resolution theorem.

This will be important for the localization theorem next time.