Lecture 9: Consequences of Additivity and Universal properties

I. Consequences of Additivity

Recall: We showed that if $\mathbb{K}(\mathscr{C})$ is an Λ-spectrum, then the additivity theorem holds. Now we will prove the converse.

Thm 1: The additivity theorem implies that $\mathbb{K}(\mathscr{C})$ is an Λ-spectrum.

This will follow from a more general result that requires some setup.

Definition. The *decalage* or *path object* of a simplicial object

$$X_\bullet : \Delta^{op} \longrightarrow \mathscr{C}$$

in a category $\mathscr{C}$ is defined by

$$(PX_\bullet)_n = X_{n+1}$$

$$d_i^{n,PX_\bullet} = d_{i+1}^{n+1,X_\bullet} \qquad 0 \le i \le n,$$

$$s_i^{n,PX_\bullet} = s_{i+1}^{n+1,X_\bullet} \qquad 0 \le i \le n.$$

Lemma: The simplicial map

$$d_0^{1,X_\bullet} : PX_\bullet \to X_0$$

induces a simplicial homotopy equivalence

$$|PX_\bullet| \to |X_0|.$$

<u>Proof</u>: First, note that $d_0^{1,X_\bullet}$ has a section $s_0^{1,X_\bullet}$ s.t. $d_0^{1,X_\bullet} s_0^{1,X_\bullet} = id_{X_0}$.

It suffices to show $s_0^{1,X_\bullet} \circ d_0^{1,X_\bullet} \simeq id_{PX_\bullet}$.

We give an explicit simplicial homotopy

$$[n] \to [1] \longmapsto (\varphi_a^* : X_{n+1} \to X_{n+1})$$

where φ_a^* is induced by the map $\varphi_a : [n+1] \to [n+1]$ defined by

$$\varphi_a(j+1) = \begin{cases} j+1 & \text{if } a(j)=0, \\ 0 & \text{if } a(j)=0 \text{ or } j=-1. \end{cases} \qquad \square$$

<u>Rmk</u>: There is a sequence of simplicial sets

$$X_1 \xrightarrow{s_0^{1,X_\bullet}} PX_\bullet \xrightarrow{d_0^{1,X_\bullet}} X_\bullet$$

Example: Let $X_\cdot = wS_\cdot^{(2)}\mathcal{C}$ regarded as a functor

$$\Delta^{op} \longrightarrow \mathrm{Wald}^{\Delta^{op}}.$$
$$[n] \longmapsto wS_n S_\cdot \mathcal{C}$$

Then this sequence is

$$wS_\cdot\mathcal{C} \longrightarrow P wS_\cdot^{(2)}\mathcal{C} \longrightarrow wS_\cdot^{(2)}\mathcal{C} \qquad (*)$$

and it factors through $wS_0 S_\cdot \mathcal{C} = *$.

Also, $|P wS_\cdot^{(2)}\mathcal{C}| \simeq |wS_0 S_\cdot\mathcal{C}| \simeq *$.

Goal: Prove that $(*)$ is a homotopy fiber sequence. Consequently, there is a homotopy equivalence

$$|wS_\cdot \mathcal{C}| \xrightarrow{\simeq} \Omega |wS_\cdot^{(2)} \mathcal{C}|$$

and replacing $\mathcal{C}$ with $S_\cdot^{(n)}\mathcal{C}$,

$$|wS_\cdot^{(n+1)}\mathcal{C}| \xrightarrow{\simeq} \Omega|wS_\cdot^{(n+2)}\mathcal{C}|$$

for all $n \geq 1$. So this would imply Theorem 1. Again, we will prove a more general statement and this needs more setup.

Def: Let $A \xrightarrow{f} B$ be a map in Wald and let $S_\cdot(A \xrightarrow{f} B)$ be defined as the pullback

$$\begin{tikzcd}
S_\cdot(A \xrightarrow{f} B) \arrow[r] \arrow[d] & P S_\cdot B \arrow[d] \\
S_\cdot A \arrow[r, "S_\cdot f"] & S_\cdot B
\end{tikzcd}$$

Unpacking this, we observe that there are pullback diagrams

$$\begin{tikzcd}
S_n(A \xrightarrow{f} B) \arrow[r] \arrow[d] & S_{n+1}B \arrow[d, "d_0^{n+1}"] \\
S_nA \arrow[r, "S_nf"] & S_nB
\end{tikzcd}$$

for each n and

$$S_n(A \xrightarrow{f} B) \cong S_nA \times_{S_nB} S_{n+1}B .$$

Also, $S_n(A \xrightarrow{f} B)$ is a Waldhausen category in an evident way.

We also have functors

$$\begin{tikzcd}
B \arrow[rr, bend left, "s_0^{1, S_\bullet B}"] \arrow[r, dashed] \arrow[d] \arrow[dr, bend left] & S_\bullet(A \xrightarrow{f} B) \arrow[r] \arrow[d] & S_{\bullet+1}B \arrow[d, "d_0^{n+1}B"] \\
S_0A \arrow[r] \arrow[d, equal] & S_\bullet A \arrow[r, "S_\bullet f"'] & S_\bullet B \\
*
\end{tikzcd}$$

So there is a sequence

$$\begin{tikzcd}
B \arrow[r] \arrow[dr, bend right] & S_\bullet(A \xrightarrow{f} B) \arrow[r] & S_\bullet A . \\
& * \arrow[ur, bend right] &
\end{tikzcd}$$

Theorem 2. The sequence

$$|N_\cdot wS_\cdot B| \longrightarrow |N_\cdot w S^{(2)}_\cdot (A \to B)| \to |N_\cdot S^{(2)}_\cdot (A)|$$

is a homotopy fiber sequence.

To prove this, we first need a lemma

Lemma: [Puppe] Let $X_{\cdot\cdot} \to Y_{\cdot\cdot} \to Z_{\cdot\cdot}$ be a sequence of bisimplicial sets so that $X_{\cdot\cdot} \to Z_{\cdot\cdot}$ is constant. Suppose that

$$|X_{\cdot n}| \to |Y_{\cdot n}| \to |Z_{\cdot n}|$$

is a homotopy fiber sequence for each n and $Z_{\cdot n}$ is connected for each n. Then

$$|X_{\cdot\cdot}| \to |Y_{\cdot\cdot}| \to |Z_{\cdot\cdot}|$$

is a homotopy fiber sequence.

Proof: See Lemma 5.2 in Waldhausen "Generalized free Products" for example.

Proof of Thm 2.

By the lemma above, it suffices to prove that

$$|wS_\bullet B| \to |wS_\bullet S_n(A \xrightarrow{f} B)| \to |wS_\bullet S_n A|$$

is a homotopy fiber sequence since $|N_\bullet wS_0 S_n A| \simeq *$.

We will use the additivity theorem to prove that this sequence is homotopy equivalent to the trivial homotopy fiber sequence; i.e

$$\begin{tikzcd}
{|wS_\bullet B|} \arrow[r] \arrow[d, equal] & {|wS_\bullet S_n(A \xrightarrow{f} B)|} \arrow[r] \arrow[d, shift right] & {|wS_\bullet S_n A|} \arrow[d, equal] \\
{|wS_\bullet B|} \arrow[r] & {|wS_\bullet B| \times |wS_\bullet S_n A|} \arrow[r] \arrow[u, shift right, "\wr"'] & {|wS_\bullet S_n A|}
\end{tikzcd}$$

An object in $S_n(A \xrightarrow{f} B)$ is a pair

$$(A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0n},\ B_{0,1} \rightarrowtail \cdots \rightarrowtail B_{0,n+1})$$

such that

$$\begin{array}{ccc}
f(A_{0,1}) \rightarrowtail & \cdots \rightarrowtail & f(A_{0n}) \\
\wr\wr & & \wr\wr \\
B_{0,2}/B_{0,1} \rightarrowtail & \cdots \rightarrowtail & B_{0,n+1}/B_{0,1}.
\end{array}$$

Let $\mathscr{C}' \subseteq S_n(A \xrightarrow{f} B)$ be the full subcategory

with objects

$$(0 \rightarrowtail 0 \rightarrowtail \cdots \rightarrowtail 0,\ B_{0,1} \overset{id}{\rightarrowtail} \cdots \overset{id}{\rightarrowtail} B_{0,n+1})$$

(Note: $B_{0,j+1}/B_{0,1} = 0 = f(0)\ \forall\ 1 \le j \le n$)

and let

$$\mathcal{E}'' \subseteq S_n(A \xrightarrow{f} B)$$

be the full subcategory with objects

$$\left(A_{0,1} \rightarrowtail \dots \rightarrowtail A_{0,n},\ 0 \rightarrowtail B_{0,2} \rightarrowtail \dots \rightarrowtail B_{0,n+1} \right)$$

$$\left(\text{so} \quad \begin{array}{ccccc} f(A_{0,1}) & \rightarrowtail & \dots & \rightarrowtail & f(A_{0,n}) \\ \wr\Vert & & & & \wr\Vert \\ B_{0,2} & \rightarrowtail & \dots & \rightarrowtail & B_{0,n+1} \end{array} \right)$$

Then clearly there are equivalences of categories

$$\mathcal{B} \rightleftarrows \mathcal{E}' \quad \text{and} \quad S_n A \rightleftarrows \mathcal{E}''.$$

Define a cofiber sequence of exact functors

$$j' \to \mathrm{id} \to j'' : S_n(A \to B) \longrightarrow S_n(A, B)$$

where j' takes values in $\mathcal{E}'$

and j'' takes values in $\mathcal{E}''$ by

$$j'(A_{\cdot\cdot}, B_{\cdot\cdot}) = (0 \rightarrowtail \cdots \rightarrowtail 0,\ B_{0,1} \overset{id}{\rightarrowtail} \cdots \overset{id}{\rightarrowtail} B_{0,1})$$

$$id(A_{\cdot\cdot}, B_{\cdot\cdot}) = (A_{\cdot\cdot}, B_{\cdot\cdot})$$

$$j''(A_{\cdot\cdot}, B_{\cdot\cdot}) = (A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n},\ 0 \rightarrowtail f(A_{0,1}) \rightarrowtail \cdots \rightarrowtail f(A_{0,n}))$$

By the additivity theorem,

$$j'_\bullet + j''_\bullet \simeq id : \underset{\substack{\parallel \\ K(S_n(A \rightarrowtail B))}}{|N_\bullet wS_\bullet S_n(A \xrightarrow{f} B)|} \longrightarrow \underset{\substack{\parallel \\ K(S_n(A \rightarrowtail B))}}{|N_\bullet wS_\bullet S_n(A \xrightarrow{f} B)|}.$$

There is an exact functor

$$S_n A \times B \underset{r}{\overset{s}{\rightleftarrows}} wS_n(A \xrightarrow{f} B)$$

$$(A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n}, B) \mapsto (A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n},\ B \rightarrowtail B \vee f(A_{0,1}) \rightarrowtail B \vee f(A_{0,2}) \rightarrowtail \cdots \rightarrowtail B \vee f(A_{0,n}))$$

$$(A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n}, B_{0,1}) \mapsfrom (A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n},\ B_{0,1} \rightarrowtail \cdots \rightarrowtail B_{0,n+1})$$

s.t. $s \circ r = id_{S_n A \times B}$.　　s.t. ...

Note that

$$r \circ s\,(A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n},\ B_{0,1} \rightarrowtail \cdots \rightarrowtail B_{0,n+1})$$

$$\shortparallel$$

$$r\,(A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n},\ B_{0,1})$$

$$\shortparallel$$

$$(A_{0,1} \rightarrowtail \cdots \rightarrowtail A_{0,n},\ B_{0,1} \rightarrowtail f(A_{0,1}) \vee B_{0,1} \rightarrowtail \cdots \rightarrowtail f(A_{0,n}) \vee B_{0,1})$$

$$\shortparallel$$

$$j' \vee j''$$

So by the additivity theorem

$$r_\bullet \circ s_\bullet \simeq \mathrm{id}_{|N_\bullet wS_\bullet S_n(A \to B)|} \qquad \square$$

II. A universal property of algebraic K-theory

Algebraic K-theory is the universal additive functor equipped with a natural transformation

$$\mathrm{ob}\,\mathscr{C} \to K(\mathscr{C}).$$

Our goal will be to make this precise.

<u>Definition</u>. A global Euler characteristic is a pair (E, χ) where E is a functor

$$E : \mathrm{Wald} \longrightarrow \mathrm{Top} \ (= \text{compactly generated Weak Hausdorff spaces})$$

and $\chi : \mathrm{ob}(-) \to E(-)$ is a natural transformation

Satisfying

1) The canonical map
$$E(\mathscr{C} \times \mathscr{D}) \to E(\mathscr{C}) \times E(\mathscr{D})$$
is a homotopy equivalence,

2) The canonical functor
$$s : \mathscr{C} \to w\mathrm{Arr}(\mathscr{C})$$
$$c \mapsto \mathrm{id}_c$$
induces a homotopy equivalence
$$E(\mathscr{C}) \simeq E(w\mathrm{Arr}(\mathscr{C}))$$

3. The Additivity theorem holds

4. The space $E(\mathscr{C})$ is a group-like H-space with multiplication

$$E(\mathscr{C}) \times E(\mathscr{C}) \underset{\simeq}{\xleftarrow{((d_0)_*, (d_2)_*)}} E(S_2\mathscr{C}) \xrightarrow{(d_1)_*} E(\mathscr{C})$$

In what sense is (E, χ) a global Euler characteristic?

Given an cofiber sequence

$$c' \rightarrowtail c'' \twoheadrightarrow c$$

in $\mathscr{C}$, naturality gives

$$\begin{tikzcd}
\mathrm{ob}(S_2\mathscr{C}) \arrow[r] \arrow[d, "(d_i)_*"'] & E(S_2\mathscr{C}) \arrow[d, "(d_i)_*"] \\
\mathrm{ob}(\mathscr{C}) \arrow[r] & E(\mathscr{C})
\end{tikzcd} \qquad 0 \le i \le 2$$

So

$$\chi_{\mathscr{C}}(c') + \chi_{\mathscr{C}}(c) = \chi_{\mathscr{C}}(c'')$$

by additivity.

Note: In fact, $\mathbb{K}(\mathcal{C})$ forms a symmetric spectrum. For this section, we write

$$K(\mathcal{C}) := \lambda^{\infty}(\mathbb{K}(\mathcal{C}))^{cf}$$

where

$$\lambda^{\infty} : Sp^{\Sigma} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} Top : \Sigma^{\infty}$$

an adjunction.

Example: $K(\mathcal{C})$ is an additive functor w/

$$\chi_{univ} : ob\,\mathcal{C} \longrightarrow K(\mathcal{C})$$

given by the adjoint

$$ob\,\mathcal{C} = Bw\mathcal{C} \longrightarrow K(\mathcal{C}) \simeq \lambda|N_{\cdot}wS_{\cdot}\mathcal{C}|$$

to the map

$$Bw\mathcal{C} \wedge S^1 = |N_{\cdot}wS_{\cdot}\mathcal{C}|_{(1)} \longrightarrow |N_{\cdot}wS_{\cdot}\mathcal{C}|$$

$$\overset{\text{1-skeleton}}{\parallel}$$

$$\coprod_{i=0}^{1}(BwS_i\mathcal{C} \times \Delta^i)/\sim .$$

<u>Def</u>: A map of global Euler characteristics is a natural transformation

$$\eta : E \Rightarrow F \quad \text{such that}$$

$$\begin{tikzcd} & ob(-) \arrow[dl, "\alpha_E"'] \arrow[dr, "\alpha_F"] & \\ E(-) \arrow[rr, "\eta"'] & & F(-) \end{tikzcd}$$

commutes.

We say $\eta : E \Rightarrow F$ is a homotopy equivalence if

$$\eta_{\mathcal{C}} : E(\mathcal{C}) \xrightarrow{\simeq} F(\mathcal{C})$$ is a

homotopy equivalence for all $\mathcal{C}$ in Wald.

Then let Eul be the category of Euler characteristics and

$$H_0(\mathrm{Eul})(E,F) = \mathrm{Eul}(E,F) \Big/ \underset{\text{htpy equiv.}}{\sim}$$

Thm [Steimle]

Algebraic K-theory (K, χ) is the initial object in $\mathrm{Ho}(\mathrm{Eul})$.

Proof sketch: Given a functor

$$F : \mathrm{Wald} \longrightarrow \mathrm{sSet}$$

define a spectrum with n-th space

$$PF_n\mathcal{C} = \operatorname*{hocolim}_{k \in \mathcal{I}} \Omega^k \Sigma^n |F(wS_\bullet^{(k)}\mathcal{C})|$$

Cat of finite sets and injective maps.

then define

$$F^{add} := \operatorname*{hocolim}_{n \in \mathbb{N}} \Omega^n PF_n\mathcal{C}$$

Ex: $ob^{ob}(\mathcal{C}) = K(\mathcal{C})$.

<u>Prop</u>: F^{add} is the additive approximation to F.

Proof sketch: Proof is similar to our proof that $\Omega^\infty K(\mathcal{C}_0)$ is additive.

To see that F^{add} is the initial additive functor equipped with a natural transformation

$$F(-) \to F^{add}(-)$$

requires using the <u>homotopy</u> category $Ho(Eul)$; i.e. any two initial objects in Eul are homotopy equivalent.