Lecture 6:

Quillen's Theorem A + B

I. Motivation

The +-construction model for algebraic K-theory is very explicit, but it has some disadvantages

(1) It is only defined for rings and not more general categories

(2) It doesn't have all the functoriality we want.

From the definition of K_0, we see that algebraic K-theory should take a category as input. Either a symmetric monoidal category, an exact category or a Waldhausen category.

To for shadow, Quillen's Q-construction for algebraic K-theory takes an exact category $\mathscr{C}$ as input, produces a category $Q\mathscr{C}$ and then we define

$$K(\mathscr{C}) = \Omega |N_\bullet(Q\mathscr{C})|$$

This recovers the +-construction model when $\mathscr{C} = P(R)$ (fin. gen proj. modules and all maps of fin. gen proj. modules.)

Def: When D is a small category,

$$BD := |N_\bullet D|.$$

So properties of $\mathcal{BD}$ will be important to the subject.

Recall: Given functors

$$\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B}$$

we can form a category $S \downarrow T$

with objects $(a \in \mathrm{ob}\,\mathcal{A}, b \in \mathrm{Ob}\,\mathcal{B}, \alpha : S(a) \to T(b))$

and morphisms $(a, b, \alpha) \to (a', b', \alpha')$ given

by $f : a \to a'$, $g : b \to b'$

$$\begin{tikzcd} S(a) \arrow[r, "S(f)"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & S(a') \arrow[d, "\alpha'"] \\ T(b) \arrow[r, "T(g)"'] & T(b') \end{tikzcd}$$

Ex:

1) $\mathcal{C}' \xrightarrow{id} \mathcal{C}' \xleftarrow{y} [0]$ $\quad \mathcal{C}'/y := id_{\mathcal{C}'} \downarrow y$

2) $[0] \xrightarrow{y} \mathcal{C}' \xleftarrow{id} \mathcal{C}'$ $\quad {}_{y\backslash}\mathcal{C}' := y \downarrow id_{\mathcal{C}'}$

3) $\mathcal{C} \xrightarrow{f} \mathcal{C}' \xleftarrow{y} [0]$ $\quad f/y := f \downarrow y$

4) $[0] \xrightarrow{y} \mathcal{C}' \xleftarrow{f} \mathcal{C}$ $\quad {}_{y\backslash}f := y \downarrow f$

II Basic Properties of classifying spaces of categories

Lemma 1: Suppose $f, g : \mathcal{C} \to D$ are functors and $\eta : f \Rightarrow g$ is a natural transformation. Then there is a homotopy

$$H : B\mathcal{C} \times I \longrightarrow BD$$

from Bf to Bg.

Proof: A natural transformation defines a functor $\mathcal{C} \times [1] \to D$

$$(c, 0) \longmapsto f(c)$$
$$(c, 1) \longmapsto g(c)$$
$$(c, 0 \to 1) \longmapsto \eta_c : f(c) \to g(c)$$

Then $N_\bullet(\mathcal{C} \times [1]) \cong N_\bullet \mathcal{C} \times \Delta^1$ and so $B(\mathcal{C} \times [1]) \cong B\mathcal{C} \times I$ by Milnor's theorem and $H : B\mathcal{C} \times I \cong B(\mathcal{C} \times [1]) \longrightarrow BD$ is a homotopy from Bf to Bg.

Lemma 2: Suppose $f: \mathcal{C} \to D$ has either a left adjoint or a right adjoint, then f induces a homotopy equivalence $Bf : B\mathcal{C} \xrightarrow{\simeq} BD$.

Proof: Suppose f has a left adjoint g without loss of generality. Then there are natural transformations

$\eta: id_D \to f\circ g$ and $\varepsilon: g\circ f \to id_{\mathcal{C}}$

inducing homotopies

$H_1: BD \times I \to BD$

from id_{BD} to $Bf \circ Bg$

and

$H_2: B\mathcal{C} \times I \to B\mathcal{C}$

from $Bg \circ Bf \simeq id_{B\mathcal{C}}$.

Lemma 3: Suppose $\mathscr{C}$ has an initial or terminal object, then $B\mathscr{C} \simeq *$.

Proof:

If $\mathscr{C}$ has an initial object 0 (resp. terminal object 1) then the functor

$$\begin{aligned} [0] &\longrightarrow \mathscr{C} \\ 0 &\longmapsto 0 \quad (\text{resp } 1) \end{aligned}$$

has a left adjoint (resp. right adjoint.)

So by Lemma 1,

$$B\mathscr{C} \simeq B[0] \simeq *.$$

III Quillen's Theorem A

Lemma: Given a bisimplicial space
$$X_{\bullet\bullet} : \Delta^{op} \times \Delta^{op} \to \mathrm{Top},$$
there are homeomorphisms
$$\big| [p] \mapsto |X_{p\bullet}| \big| \cong \big| [q] \mapsto |X_{\bullet q}| \big|$$
$$\cong \big| [n] \mapsto X_{nn} \big| .$$

Definition: Let $Tw(f)$ be the category with objects

$$\mathrm{ob}\, Tw(f) = \mathrm{id}_{\mathcal{C}'} \downarrow f \ni (a \in \mathrm{ob}\,\mathcal{C}', b \in \mathrm{ob}\,\mathcal{C}, \alpha : a \to f(b))$$

and morphisms

$$(a, b, \alpha : a \to f(b)) \longrightarrow (a', b', \alpha' : a' \to f(b'))$$

given by

$$\left(u : a \to a',\ v : b' \to b,\ \begin{array}{ccc} a & \xrightarrow{\alpha} & f(b) \\ {\scriptstyle u}\downarrow & \circlearrowright & \uparrow{\scriptstyle f(v)} \\ a' & \xrightarrow[\alpha']{} & f(b') \end{array} \right)$$

Theorem A

Suppose $f: \mathcal{C} \to \mathcal{C}'$ is a functor and suppose $\forall\, Y \in \mathrm{ob}\,\mathcal{C}'$ there is a homotopy equivalence $\quad B\, Y\backslash f \simeq *$

then $Bf : B\mathcal{C} \to B\mathcal{C}'$

is a homotopy equivalence.

Proof: Consider the span

$$(\mathcal{C}')^{op} \xleftarrow{\pi_2} \mathrm{Arr}(f) \xrightarrow{\pi_1} \mathcal{C}$$
$$b \longmapsfrom (a, b, \alpha : a \to f(b)) \longmapsto a$$

of categories. We form a bisimplicial set $T_{\bullet,\bullet}$ w/ (p,q)-simplices

$$(y_p \to y_{p-1} \to \dots \to y_0 \to f(x_0),\ x_0 \to x_1 \to \dots \to x_q)$$

where face maps in p & q direction are given by composition and degeneracies are given by inserting identities.

Then

$$T_{pp} = (y_p \to y_{p-1} \to \cdots \to y_0 \to f(x_0),\ x_0 \to x_1 \to \cdots \to x_q)$$

which is the same data as a triple

$$\left(y_p \to y_{p-1} \to \cdots \to y_0,\ x_0 \to x_1 \to \cdots \to x_q,\ \begin{tikzcd} y_p \arrow[r] \arrow[d] & y_{p-1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & y_0 \arrow[d] \\ f(x_p) & \cdots \arrow[l] & f(x_1) \arrow[l] & f(x_0) \arrow[l] \end{tikzcd} \right)$$

or in other words $|[n] \to T_{nn}| \cong |Tw(f)|$

There are also natural maps

$$N_p(\mathcal{C}')^{op} \longleftarrow T_{p,q} \longrightarrow N_q\mathcal{C}$$

of bisimplicial sets.

Taking geom. realization in the p-direction we have a map of simplicial spaces

$$\coprod_{x_0 \to x_1 \to \cdots \to x_q} B(f(x_0)\backslash\mathcal{C}')^{op} \longrightarrow \coprod_{x_0 \to x_1 \to \cdots \to x_q} * = N_q\mathcal{C}$$

And $f(x_0)\backslash\mathscr{C}'$ has an initial object so

$$B(f(x_0)\backslash\mathscr{C}')^{op} \cong B_{f(x_0)\backslash}\mathscr{C}' \simeq *$$

Since both sides are proper simplicial spaces, this levelwise weak equivalence induces a weak equivalence

$$|[q] \longmapsto |T_{p,\mathscr{C}}|| \xrightarrow{\sim} |N_\bullet\mathscr{C}|,$$

but since both sides are CW complexes this is a homotopy equivalence by Whitehead's theorem.

So

$$|Tw(f)| \simeq |N_\bullet\mathscr{C}|.$$

Considering the realization of

of the map $N_p(\mathscr{C}')^{op} \leftarrow T_{p,q}$

in the q-direction produces a map

$$\coprod_{y_p \to y_{p-1} \to \cdots \to y_0} B_{y_0} \backslash f \longrightarrow \coprod_{y_p \to y_{p-1} \to \cdots \to y_0} * = N_p((\mathscr{C}')^{op})$$

But, by assumption

$$B_{y_0} \backslash f \simeq * \quad \text{for all } y_0 \in \mathrm{ob}\, \mathscr{C}'.$$

So by the same considerations as before

$$B(\mathscr{C}')^{op} \simeq B(Tw\, f) .$$

Finally, we consider the diagram

$$\begin{tikzcd}
\mathcal{C}'^{op} \arrow[d, "id"] & Tw(f) \arrow[l] \arrow[r] \arrow[d, "f'"] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{C}'^{op} & Tw(id_{\mathcal{C}'}) \arrow[l] \arrow[r] & \mathcal{C}'
\end{tikzcd}$$

Then we have a commuting diagram

$$\begin{tikzcd}
B(\mathcal{C}')^{op} \arrow[d, "\simeq"'] & BTw f \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & B\mathcal{C} \arrow[d, "Bf"] \\
B(\mathcal{C}')^{op} & BTw(id_{\mathcal{C}'}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & B\mathcal{C}'
\end{tikzcd}$$

where each arrow decorated by $\simeq$ is a homotopy equivalence.

So Bf is a homotopy equivalence.

Special case:

Given a functor $f: \mathscr{C} \to \mathscr{D}$

let $f^{-1}(y)$ be the full subcategory of $\mathscr{C}$ w/ objects $x \in \mathrm{ob}\,\mathscr{C}$ s.t. $f(x) = y$.

We say f is pre-fibered (resp. pre-cofibered) if the canonical functor

$$f^{-1}(y) \to y \backslash f$$

$$x \longmapsto (x, y, y \xrightarrow{\mathrm{id}_y} f(x))$$

has a left adjoint $(x,v) \mapsto v^* x$ called base change (resp. right adjoint $(x,v) \mapsto v_* x$ called cobase change)

So $v^*: f^{-1}(y') \to f^{-1}(y)$ (resp. $v_*: f^{-1}(y') \to f^{-1}(y)$) are functors. We say f is fibered (resp. cofibered) if $v^* \circ w^* \xrightarrow{\cong} (v \circ w)^*$ (resp. $v_* \circ w_* \xleftarrow{\cong} (v \circ w)_*$)

Cor. If $f : \mathscr{C} \to \mathcal{D}$ is pre-fibered (or pre-cofibered) and $Bf^{-1}(y) \simeq *$ for all $y \in \mathrm{ob}\,\mathcal{D}$ then $Bf : B\mathscr{C} \xrightarrow{\simeq} B\mathcal{D}$ is a homotopy equivalence.

Proof: By Lemma 2,

$$Bf^{-1}(y) \simeq By\backslash f$$

so the result follows by Theorem A.

IV Quillen's Theorem B

We now prove a more general result that measures the failure of the map $Bf : B\mathcal{C} \to B\mathcal{C}'$ to be a weak equivalence.

<u>Def</u>: We say

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"] & W \end{tikzcd}$$

is a htpy pullback

the map $X \to hPB$ is a weak equivalence ($\pi_k X \xrightarrow{\cong} \pi_k hPB$ $\forall k \geq 0$)

where hPB is the pullback

$$\begin{tikzcd} hPB \arrow[r] \arrow[d] & W^I \arrow[d] \\ Z \times Y \arrow[r, "f \times g"'] & W \times W \end{tikzcd}$$

$I \to W$

$\downarrow$

$\{0,1\} \hookrightarrow I \to W$

$W^{\{0,1\}} = W \times W$

If $z = *$, we say $y \xrightarrow{g} w$

is a quasi-fibration if

$$g^{-1}(w) \simeq hPB =: \mathrm{Fib}(g).$$

<u>Theorem B</u> Let $f : \mathcal{C} \to \mathcal{C}'$

be a functor such that for every map $v : y \to y'$ the induced functor $v \backslash f : y' \backslash f \to y \backslash f$ induces a homotopy equivalence $Bv \backslash f : B(y' \backslash f) \to By \backslash f$.

Then for every $y \in \mathcal{C}'$, there is a homotopy pullback

$$\begin{tikzcd}
By \backslash f \arrow[r] \arrow[d] & B\mathcal{C} \arrow[d, "Bf"] \\
* \simeq By \backslash \mathcal{C}' \arrow[r] & B\mathcal{C}'
\end{tikzcd}$$

Consequently, there is a long exact sequence in homotopy

$$\to \pi_{i+1}(B_y \backslash f, \bar{x}) \to \pi_{i+1}(B\mathscr{C}, x) \to \pi_{i+1}(B\mathscr{C}', y) \to \pi_i(B_y\backslash f, \bar{x}) \to \pi_i(B\mathscr{C}, x) \to \pi_i(B\mathscr{C}', y) \to \cdots$$

where $x \in \text{ob}\, f^{-1}(y)$ and
$\bar{x} = (x, id_y : f(x) \to y)$

We will prove Theorem B assuming the following lemma.

<u>Lemma</u>: Under the hypotheses of Theorem B, $B\pi_2 : BTw(f) \to B(\mathscr{C}')^{op}$ is a quasi-fibration for any functor $f: \mathscr{C} \to \mathscr{C}'$.

Proof of Theorem B assuming the lemma

Consider the diagram

$$\begin{tikzcd}
y\backslash f \arrow[r, "{(x,\, y\to f(x)) \mapsto (x,\, y,\, y\to f(x))}"] \arrow[d] & Tw(f) \arrow[r] \arrow[d, "f'"] & \mathcal{Y} \arrow[d, "f"] \\
y\backslash \mathcal{Y}' \arrow[r] \arrow[d] & Tw(id_{\mathcal{Y}'}) \arrow[r] \arrow[d] & \mathcal{Y}' \\
{[0]} \arrow[r] & \mathcal{Y}' &
\end{tikzcd}$$

So the diagram

$$\begin{tikzcd}
B y\backslash f \arrow[r] \arrow[d] & B\mathcal{Y} \arrow[d, "f"] \\
* \simeq B y\backslash \mathcal{Y}' \arrow[r, "h"] & B\mathcal{Y}'
\end{tikzcd}$$

is a homotopy pullback.

Corollary:

Suppose $f: \mathscr{C} \to \mathscr{C}'$ is pre-fibered (resp. pre-cofibered) and for all $v: y \to y'$ in $\mathscr{C}'$ $(I \to B\mathscr{C}')$ there is a homotopy equivalence

$$B v^*: Bf^{-1}(y) \xrightarrow{\simeq} Bf^{-1}(y')$$

$$\left(\text{resp. } B v_* : Bf^{-1}(y) \xrightarrow{\simeq} Bf^{-1}(y')\right)$$

then $\quad Bf^{-1}(y) \simeq \mathrm{Fib}(Bf)$

and we have a long exact sequence in homotopy

$$\pi_{i+1}(Bf^{-1}(y), x) \to \pi_{i+1}(B\mathscr{C}, x) \to \pi_{i+1}(B\mathscr{C}', y) \to$$
$$\to \pi_i(Bf^{-1}(y), x) \to \pi_i(B\mathscr{C}, x) \to \pi_i(B\mathscr{C}', y) \to \cdots$$

for $x \in \mathrm{ob}\, f^{-1}(y)$.