# Lecture 7 : Covering spaces and classifying spaces of categories.

# I. Quasi-Fibrations and Quillen's theorem B

To prove the key lemma finishing the proof of Quillen's theorem B, we first need three lemmas about ________________ .

**Lemma 1**: Let $p: E \to B$ be a continuous map and let $U, V \subseteq B$ be subspaces such that $U \cap V \neq \emptyset$ and $U \cup V = B$. If $p|_{p^{-1}(U)}$, $p|_{p^{-1}(V)}$, and $p|_{p^{-1}(U \cap V)}$ are quasi-fibrations, then $p$ is a quasi-fibration.

( $\underset{\text{based map}}{p: E \to B}$ quasifibration $\underset{\text{is a weak equivalence}}{p^{-1}(b) \to \mathrm{Fib}(p,b)}$ )

**Lemma 2** Let $p: E \to B$ be a continuous map onto $B$, let $B' \subset B$ be a subspace and let $E' = p^{-1}(B')$. Suppose there is a fiber preserving deformation

$$\begin{tikzcd} E \arrow[r, "D_t"] \arrow[d] & E \arrow[d] \\ B \arrow[r, "d_t"] & B \end{tikzcd} \quad t \in [0,1]$$

such that

$$D_0 = id_E, \; d_0 = id_B, \; D_t(E') = E',$$

$$d_t(B') = B' \text{ and } D_1(E) \subset E'$$

and $d_1(B) \subset B'$. Additionally, assume that $\underline{p^{-1}(b) \longrightarrow p^{-1}(d_1(b))}$ is a weak equivalence for all $b \in B$, then $p$ is a quasi-fibration.

**Lemma 3:** Let $p: E \to B$ be a continuous map. Assume $B$ is a CW complex with n-skeleton $B_i$ and assume $\underline{p|_{p^{-1}(B_i)}}$ is a quasi-fib. $\star$ for all $i \geq 0$, then $p$ is a quasi-fibration.

Proof: Any compact subset of $B$ lies in some $B_i$, so any compact subset of $E$ lies in some $E_i = p^{-1}(B_i)$. Consequently, for any $x \in B_i$, $y \in p^{-1}(x)$

$$\pi_n(E, p^{-1}(x); y) \cong \operatorname{colim}_j \pi_n(E_j, p^{-1}(x); y)$$

$$\underline{\text{by } \star} \qquad \cong \operatorname{colim}_j \pi_n(B_j; x)$$

$$\cong \pi_n(B, x). \qquad \square$$

**Lemma** (Technical lemma for Thm B)

Suppose $I$ is a small category and $X : I \to \mathrm{Top}$ is a functor

Define a simplicial space $X_I$ with $n$-simplices

$$[n] \longmapsto \coprod_{i_0 \to \cdots \to i_n \in N_n(I)} X(i_0).$$

Assume $X(i) \to X(j)$ is a weak equivalence for all $i \to j$ in $I$.

Then the canonical map of simplicial spaces

$$(X_I)_\bullet \longrightarrow N_\bullet I$$

induces a quasi-fibration

$$|X_I| \longrightarrow BI.$$

Proof: By Lemma 3, it suffices to check that

$$|X_{\underline{I}}|_p \xrightarrow{\pi} (\mathbb{B}\underline{I})_p$$

is a quasi-fibration for all $p \geq 0$. On 0-cells this is a trivial quasi-fibration

We then induct up the cells.

Assume that

$$|X_{\underline{I}}|_{p-1} \longrightarrow (\mathbb{B}\underline{I})_{p-1}$$

is a quasi-fibration.

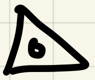
We then consider the map of p-shafts

$$\begin{tikzcd}
\coprod\limits_{i_0\to\cdots\to i_p \in ND_p(N_\bullet I)} \partial|\Delta^p| \times X_{i_0} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & {|X_I|_{p-1}} \arrow[d, hook] \\
\coprod\limits_{i_0\to\cdots\to i_p \in ND_p(N_\bullet I)} |\Delta^p| \times X_{i_0} \arrow[r] & {|X_I|_p \supseteq p^{-1}U}
\end{tikzcd}$$

$$\Big\downarrow$$

$$\begin{tikzcd}
\coprod\limits_{\substack{i_0\to\cdots\to i_p \\ \in \\ ND_p(N_\bullet I)}} \partial|\Delta^p| \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & BI_{p-1} \arrow[d] & p^{-1}(i) = X_i \\
\coprod\limits_{\substack{i_0\to\cdots\to i_p \\ \in \\ ND_p(N_\bullet I)}} |\Delta^p| \arrow[r] & BI_p \supseteq U & \exists\, i \text{ object of } I
\end{tikzcd}$$

$$BI_0 = \coprod_{i \in \pi_0 BI} \circledast$$

Then let $U$ be $BI_p$ with the barycenter (Ex: ) of each $p$-cell removed and

let $V = BI_p - BI_{p-1}$. Then

$U \cup V = BI_p$ and $U \cap V \neq \emptyset$.

Since $BI_p - BI_{p-1} \cong \coprod_{i_0 \to \cdots \to i_p} |\Delta^p| - \partial|\Delta^p|$

and the map

$$\begin{tikzcd}
\coprod_{i_0 \to \cdots \to i_p} \left( (|\Delta^p| \times X_{i_0}) - (\partial|\Delta^p| \times X_{i_0}) \right) \arrow[d] \arrow[r, phantom, "\cong"] & \coprod_{i_0 \to \cdots \to i_p} (|\Delta^p| - \partial|\Delta^p|) \times X_{i_0} \\
\coprod_{i_0 \to \cdots \to i_p} |\Delta^p| - \partial|\Delta^p| &
\end{tikzcd}$$

$(\pi : |XI|_p \to BI_p$

quasi-

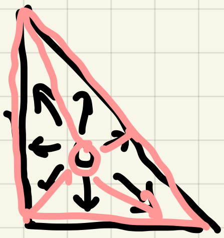
is a trivial fibration, we know $\pi|_V$

and $\pi|_{U \cap V}$ are quasi-fibrations.

It therefore suffices to check that $\pi|_{\pi^{-1}(U)}$ is a quasifibration by Lemma 1.

We construct a fiber preserving deformation

$$\begin{tikzcd}
p^{-1}(U) \arrow[r] \arrow[d, "D_t"'] & U \arrow[d, "d_t"] \\
p^{-1}(U) \arrow[r] & U
\end{tikzcd}$$



w/ $d_1(U) \subseteq (\mathbb{B}I)_{p-1}$

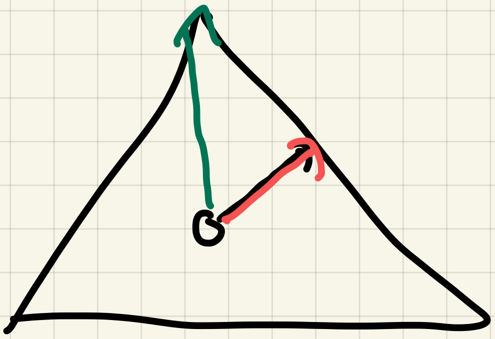
by considering a radial deformation of each p-cell onto the boundary.

We need to show

$$\pi|_{p^{-1}(U)}^{-1}(x) \longrightarrow \pi|_{p^{-1}(U)}^{-1}(\overset{x'}{\overset{\shortparallel}{d_1(x)}})$$

is a weak equivalence for all $x \in U$ by Lemma 2.

Our deformation takes $x$



to some elt. in a lower cell

$$\coprod_{i_{j_0}\to i_{j_1}\to\cdots\to i_{j_q}} |\Delta^q| \ni \overset{x'}{\overset{\shortparallel}{d_1(x)}}$$

$ND_q(N_\bullet I) \nearrow$

for $\{j_0, \dots, j_q\} \in \{0, \dots p\}$

Thus, $\pi|^{-1}_{p^{-1}(U)}(x) = X(i_0)$ and

$$\nearrow \pi|^{-1}_{p^{-1}(U)}(\overset{x'}{\overset{\shortparallel}{d_1(x)}}) = X(i_{j_0})$$

and the map $\qquad x \in BI_\bullet$

$$X_{i_0} = p|_U^{-1}(x) \to Fib(p|_U, x) = X_{i_{j_0}}$$

is induced by the map $\quad i_0 \to i_{j_0}$

in $I$ which is a weak equivalence

by assumption. □

<u>Corollary</u>: Given a functor $\mathcal{C} \xrightarrow{f} \mathcal{C}'$,

then $\underline{\overset{|[p] \mapsto T_{p,p}|}{\overset{\shortparallel}{BTW(f)}} \to B(\mathcal{C}')^{op}}$

is a quasi-fibration whenever

$v\backslash f : y\backslash f \to y'\backslash f$ induces

a weak equivalence

$$\underline{B y\backslash f \to y'\backslash f}$$

for all $v : y \to y'$ in $\mathcal{C}'$.

Proof:

Let $X : (\mathcal{C}')^{op} \to Top$ be

$X(y) = B y\backslash f$. Then apply the

previous lemma.

This completes the proof of Theorem B.

# II Covering spaces and the classifying space of a category

Let $\mathcal{C}$ be a small category. By a morphism inverting functor $F : \mathcal{C} \to \mathrm{Set}$ we mean a functor that sends all maps $v : y \to y'$ in $\mathcal{C}$ to isomorphism.

Def: $\mathcal{C}[\mathrm{Arr}(\mathcal{C})^{-1}]$ is a category satisfying the universal property

$$\mathrm{Fun}(\mathcal{C}[\mathrm{Arr}(\mathcal{C})^{-1}], \mathrm{Set}) \cong \mathrm{Fun}'(\mathcal{C}, \mathrm{Set})$$

where $\mathrm{Fun}'(\mathcal{C}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$

is the full subcategory of morphism inverting functors.

When $\mathcal{C}$ is a small category,

$\mathcal{C}[\mathrm{Arr}(\mathcal{C})^{-1}]$ is a groupoid

(a small category where all morphisms are invertible).

Ex: $\mathcal{C}$ has a single object, then

$\mathcal{C}[\mathrm{Arr}(\mathcal{C}^{-1})] = G$ is a group regarded

as a category with one object.

In this case $G = \mathcal{C}^{sp}$ where $\mathcal{C}$

is regarded as a monoid.

A functor $G = \mathcal{C}[\mathrm{Arr}(\mathcal{C}^{-1})] \to \mathrm{Set}$

is then a G-set.

**Thm**: There is an equivalence of categories

$$\mathrm{Fun}(\underbrace{\mathcal{C}[\mathrm{Arr}^{-1}]}_{\text{groupoid}}, \mathrm{Set})$$

$$\|$$

$$\mathrm{Cov}(B\mathcal{C}) \rightleftarrows \mathrm{Fun}'(\mathcal{C}, \mathrm{Set}).$$

**Proof**. We first specify the functors in each direction.

Given a covering $E \xrightarrow{p} B\mathcal{C}$, we define a functor $E : \mathcal{C} \to \mathrm{Set}$ by

$$E_x = p^{-1}(x)$$

$$E_{(x \to y)} = p^{-1}(x) \to p^{-1}(y).$$

By hypothesis, $E_{x \to y}$ is an iso of sets for all $x \to y$ in $\mathcal{C}$.

$$\begin{tikzcd} E \arrow[d] & \\ B\mathscr{C} \arrow[r, mapsto] & E_{(-)} : \mathscr{C} \to \mathrm{Set} \end{tikzcd}$$

.

Now given a morphism inverting functor

$$f : \mathscr{C} \to \mathrm{Set}$$

we can post compose $f : \mathscr{C} \to \mathrm{Set} \hookrightarrow \mathrm{Cat}$ with the inclusion of Set in the category of small categories, sending a set $S$ to the category $\underline{S}$ w/ $\mathrm{ob}\,\underline{S} = S$ $\mathrm{Arr}\,\underline{S} = S = \{id_s\}$.

We form the category $[0]\backslash f$ which is the comma category of

$$[0] \xrightarrow{[0]} \mathrm{Cat} \hookleftarrow \mathrm{Set} \xleftarrow{f} \mathscr{C} .$$

$$\mathrm{ob}([0]\backslash f) = (c \in \mathrm{ob}\,\mathscr{C},\ * \to f(c))$$

$$\begin{tikzcd} c \arrow[r, "v"] & c' \end{tikzcd} \in \mathscr{C}$$

$$(c, \alpha : * \to f(c)) \to (c', \alpha' : * \to f(c'))$$

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ f(c) \arrow[rr] & & f(c') \end{tikzcd}$$

**Lemma.** The map

$$B_{(0)}\int f \to B\mathcal{C}$$

induced by the functor

$$(c, d : * \to f(c)) \longmapsto c$$

$$\left( \begin{tikzcd} c \arrow[d] \\ c' \end{tikzcd} , \begin{tikzcd} & f(c) \arrow[d] \\ * \arrow[ur] \arrow[r] & f(c') \end{tikzcd} \right) \longmapsto \begin{tikzcd} c \arrow[d] \\ c' \end{tikzcd}$$

is a covering.

Proof: Exercise (see Appendix Gabriel–Zisman)

Note: Both of these constructions are easily seen to be functorial.

$$\left( id \xrightarrow{\cong} G \circ F \qquad F \circ G \xrightarrow{\cong} id \right)$$

We check that there is a natural iso

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] & B_{[0]}\backslash E_{-} \arrow[d] \\
B\mathscr{C} \arrow[r, "="'] & B\mathscr{C}
\end{tikzcd}$$

and a natural iso

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d, "(B_{[0]}\backslash F)_{-}"'] & \mathscr{C} \arrow[d, "F"] \\
\mathrm{Set} \arrow[r, "="'] & \mathrm{Set}
\end{tikzcd}$$

.

First, define

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] & & B_{[0]}\backslash E_{-} \arrow[dl] \\
& B\mathscr{C} &
\end{tikzcd}$$

$$\{e \in E_c\} \cong \{* \to E_c\}.$$

$$e \in E_c \longrightarrow (c, * \xrightarrow{e} E_c)$$

Similarly, define

$$\begin{tikzcd} \mathcal{C} \arrow[r, "id"] \arrow[dr] & \mathcal{C} \arrow[d, "F"] \\ & Set \end{tikzcd}$$

$$c \mapsto (B_{[0]} \backslash F)_c = \{ * \to F(c) \}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

$$\qquad\qquad\qquad\qquad\qquad\qquad F(c)$$

then the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "id"] \arrow[d, "(B_{[0]} \backslash f)_-"'] & \mathcal{C} \arrow[d, "f"] \\ Set \arrow[r, "id"'] & Set \end{tikzcd}$$

So

$$(B_{[0]} f)_- = f \, .$$

Cor. There is an isomorphism

$$\pi_1(B\mathcal{C}, c) \cong \underline{\mathrm{Aut}_{\mathcal{C}[\mathrm{Arr}^{-1}]}(c)} .$$

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We will use this to identify

$\pi_0(\Omega BQP(R)) = \pi_1(BQP(R))$

with $\underline{K_0(R)}$ where

$P(R)$ = finitely generated proj & all maps of finitely generated proj.

$\underline{K_0(R) \times BGL(R)^+}$

$\|$

$K(R) = \underline{\Omega B Q P R}$

exact category ↦ small category

# III. $K_0$ of an exact category

**Def:** An Ab-enriched category

is a category $\mathscr{C}$ w/

$$\mathrm{Hom}_{\mathscr{C}}(c,c') \in \mathrm{Ab}$$

for all $c, c' \in \mathrm{ob}\,\mathscr{C}$.

We say an Ab-enriched category is an additive category if it is closed under finite coproducts. (Consequently, it has a zero object 0 and all finite biproducts denoted $\oplus$.)

**Def**: We say an additive category is an **abelian category** if it is closed under finite limits & colimits and every map $f: A \to B$ factors as

$$A \xrightarrow[\text{epi}]{p} \operatorname{coker}(\ker f) \xrightarrow{\cong} \ker(\operatorname{coker} f) \xrightarrow[\text{mono}]{i} B$$

So we can write

$$A \to \operatorname{im} f \to B.$$

**Def:** An exact category $\mathcal{A}$ is an additive sub category of an abelian category $\mathcal{C}$. such that whenever

$$0 \to X \to Y \to Z \to 0$$

is an exact in $\mathcal{C}$ and $X, Z \in \operatorname{ob} \mathcal{A}$ then $Y \in \operatorname{ob} \mathcal{A}$.

Note: we don't have all kernels + cokernels

Let $E$ be be the class of sequences

$$0 \to X \rightarrowtail Y \twoheadrightarrow Z \to 0 \qquad \circledast$$

in $\mathcal{A}$ where $\circledast$ is exact in $\mathcal{C}$.

In this, case we say $X \rightarrowtail Y$ is an admissible monomorphism and $Y \twoheadrightarrow Z$ is an admissible epimorphism.

Examples:

- Any abelian category is an exact category

$\subseteq \mathrm{Mod}_R$

- $P(R)$ finitely generated proj. $R$-modules (all exact sequences split.)

<

$\subseteq \mathrm{Mod}_R$

- $M(R)$ finitely generated $R$-modules

$\subseteq$ families of vector spaces parametrized by $X$

- $VB(X)$ vector bundles over $X$

  $X$ a space

$\subseteq \mathcal{O}_X$-modules

- $VB(X)$ = algebraic vector bundles

  $X$ a scheme

**Def**: Given a small exact category $\mathcal{A}$ define

$$K_0(\mathcal{A}) = \frac{\mathbb{Z}[\text{iso}\,\mathcal{A}]}{\left( [B]=[A]+[C] \text{ whenever } 0 \to A \rightarrowtail B \twoheadrightarrow C \to 0 \in E \right)}$$

Ex: $\mathcal{A}$ split exact category

is an exact category $\mathcal{A}$ in which every exact sequence splits

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d, phantom, sloped, "\cong"] & C \arrow[r] \arrow[l, bend right] & 0 \\ & & A\oplus C & & \end{tikzcd}$$

.

In this case,

$$K_0(\mathcal{A}) = K_0^{\oplus}(\mathcal{A})$$

Ex:

$$\begin{array}{ccc} K_0(P(R)) & = & K_0^{\oplus}(P(R)) \\ \parallel & & \parallel \\ \dfrac{\mathbb{Z}[\mathrm{iso}\,P(R)]}{\left([A]+[C]=\underset{\overset{\parallel}{[A\oplus C]}}{[B]}\right)} & & \dfrac{\mathbb{Z}[\mathrm{iso}\,P(R)]}{\left([A]+[C]=[A\oplus C]\right)} \end{array}$$