Lecture 9 : Proof of the Additivity theorem

I. Recollections

Let $\mathcal{C}$ be a Waldhausen category

$S_2\mathcal{C}$ $\quad$ $\mathrm{ob}\, S_2\mathcal{C} \ni A \rightarrowtail B \twoheadrightarrow C$

$S_2\mathcal{C}(A \rightarrowtail B \twoheadrightarrow C, A' \rightarrowtail B' \twoheadrightarrow C') \ni$
$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

$cS_2\mathcal{C}(A \rightarrowtail B \twoheadrightarrow C, A' \rightarrowtail B' \twoheadrightarrow C') \ni$
$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, tail] \\ A' \arrow[r, tail] & B' \arrow[r] & C' \end{tikzcd}$$

$wS_2\mathcal{C}(A \rightarrowtail B \twoheadrightarrow C, A' \rightarrowtail B' \twoheadrightarrow C') \ni$
$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\wr"'] & B \arrow[r, two heads] \arrow[d, "\wr"'] & C \arrow[d, "\wr"] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

More generally, there is a functor

$$S_\bullet : \mathrm{Wald} \longrightarrow \mathrm{Wald}^{\Delta^{op}}$$

And, we define

$$K(\mathcal{C}) := |N_\bullet w S_\bullet \mathcal{C}|.$$

<u>Thm</u> [Additivity] The exact functor
$$(d_0, d_2) : S_2\mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$$
induces a homotopy equivalence
$$K(S_2\mathcal{C}) \xrightarrow{\simeq} K(\mathcal{C}) \times K(\mathcal{C}).$$

Equivalent formulations:

1) Given Waldhausen categories $\mathcal{A}$, $\mathcal{C}$, & $\mathcal{B}$ and fully faithful functors $\mathcal{A} \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{B}$. The functor
$$(d_0, d_1) : \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \longrightarrow \mathcal{A} \times \mathcal{B}$$
induces a homotopy equivalence
$$K(\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})) \to K(\mathcal{A}) \times K(\mathcal{B})$$

2) There is a homotopy equivalence
$$(d_0)_* \vee (d_2)_* \simeq (d_1)_* : K(S_2 \mathcal{C}) \longrightarrow K(\mathcal{C}).$$

3) For any cofiber sequence
$$F' \rightarrowtail F \twoheadrightarrow F'' : \mathcal{C}' \longrightarrow \mathcal{C}$$
of exact functors between Waldhausen categories there is a homotopy equivalence.
$$F'_* \vee F''_* \simeq F_* : K(\mathcal{C}') \to K(\mathcal{C})$$

4) The spectrum $\mathbb{K}(\mathcal{C}) = \{ K(\mathcal{C})_n , \sigma_n : K(\mathcal{C})_n \to \Omega K(\mathcal{C})_{n+1} \}$ is an Ω-spectrum; i.e. the maps
$$K(\mathcal{C})_n := |N_\bullet w S_\bullet^{(n)} \mathcal{C}| \to \Omega |N_\bullet w S_\bullet^{(n+1)} \mathcal{C}| =: \Omega K(\mathcal{C})_{n+1}$$
are homotopy equivalences.

Today, we will prove the additivity theorem.

II. Proof of the Additivity theorem

We will first consider the special case where $w\mathcal{C}$ is the minimal choice; i.e. weak equivalences are exactly the isomorphisms in $\mathcal{C}$.

First, we show that this special case can be reduced further.

Def: $s_n\mathcal{C} := \operatorname{ob} S_n\mathcal{C}$.

Lemma: An exact functor $f: \mathcal{C} \to \mathcal{C}'$ between categories with cofibrations induces a map

$$f_\bullet : s_\bullet\mathcal{C} \to s_\bullet\mathcal{C}'$$

and a natural isomorphism $\eta: f \overset{\cong}{\Rightarrow} g$ between two such functors induces a homotopy between $f_\bullet$ and $g_\bullet$. In particular, an exact equivalence of categories induces a homotopy equivalence.

Proof: Exercise

Hint: A simplicial homotopy $X_\bullet \times \Delta^1 \to Y_\bullet$ is equivalent data to a natural transformation of functors

$$\begin{array}{ccccc} \Delta/_{[1]}^{op} & \to & \Delta^{op} & \longrightarrow & D \\ {[n]} \to [1] & \mapsto & [n] & \longrightarrow & X_n \\ & & \Downarrow & & \\ {[n]} \to [1] & \mapsto & [n] & \longrightarrow & Y_n \end{array}$$

Cor: There is a homotopy equivalence

$$|s_\bullet \mathcal{C}| \simeq |N_\bullet \mathrm{iso}\, S_\bullet \mathcal{C}|$$

Proof: The functor

$$\mathrm{ob}\,\mathcal{C} \longrightarrow \mathrm{iso}\,\mathcal{C}$$

is an exact functor of categories with cofibrations and it is an equivalence of categories.

Therefore, the special case $wS_\bullet\mathcal{C} = \mathrm{iso}\,S_\bullet\mathcal{C}$ of the additivity theorem follows from

Proposition (Additivity special case)

The exact functor $(d_0, d_2) : S_2\mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ induces a homotopy equivalence

$$((d_0)_*, (d_2)_*) : |s_\bullet S_2\mathcal{C}| \xrightarrow{\simeq} |s_\bullet\mathcal{C}| \times |s_\bullet\mathcal{C}|.$$

Next, we show why the additivity theorem reduces to this special case.

<u>Proof</u> (Previous proposition implies additivity)

We define a full subcategory of

$$C([n], \mathcal{C}, w\mathcal{C}) \subseteq \mathrm{Cat}([n], \mathcal{C})$$

whose objects take values in $w\mathcal{C}$.

This clearly forms a simplicial Waldhausen category

$$C([n], \mathcal{C}, w\mathcal{C}) : \Delta^{op} \longrightarrow \mathrm{Cat}.$$

We note that there is an isomorphism

$$N_p w S_q (S_2 \mathcal{C}) \cong s_q(S_2 C([p], \mathcal{C}, w\mathcal{C}))$$

of bisimplicial sets and also

$$N_p w S_q \mathcal{C} \cong s_q C([p], \mathcal{C}, w\mathcal{C})$$

So applying the previous proposition to $C([p], \mathcal{C}, w\mathcal{C})$ implies the additivity theorem. $\square$

It therefore suffices to prove the proposition. This requires three lemmas.

<u>Lemma A</u> Let $y \in Y_n$ and $f: X \to Y$ a map of simplicial sets then let $f/_{(n,y)}$ denote the pullback

$$\begin{tikzcd} f/_{(n,y)} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "y"'] & Y \end{tikzcd} \quad (*)$$

If $f/_{(n,y)}$ is contractible for every (n,y), then $X \to Y$ is a homotopy equivalence.

<u>Lemma B</u> If for every $\alpha: [m] \to [n]$ in Δ and every $y \in Y_n$ the induced map

$$f/_{(m, \alpha^* y)} \longrightarrow f/_{(n,y)}$$

is a homotopy equivalence, then for every (n,y) the pullback $(*)$ is a homotopy pullback.

<u>Proof</u>: Recall that given a simplicial set Y we can form the category

$$\Delta/Y \qquad \begin{array}{l} \mathrm{ob}\,\Delta/Y : \Delta^n \to Y \\ \Delta/Y(\Delta^m \xrightarrow{y'} Y, \Delta^n \xrightarrow{y} Y) \end{array} \qquad \begin{tikzcd} \Delta^m \arrow[rr, "\alpha"] \arrow[dr, "y'"'] & & \Delta^n \arrow[dl, "y"] \\ & Y & \end{tikzcd}$$

i.e. $\alpha^* y = y'$.

This defines a functor

$$\Delta/- : \mathrm{Set}^{\Delta^{op}} \to \mathrm{Cat}.$$

This functor preserves pullbacks, so

$$\Delta/(f/(n,y)) \cong (\Delta/f)/(n,y)$$

We therefore apply $\Delta/-$ to the diagram

$$\begin{tikzcd} f/(n,y) \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

and then apply Quillen's theorem A and Quillen's theorem B.

Lemma [Technical lemma]

The exact functor $d_0 : S_2\mathscr{C} \to \mathscr{C}$

induces a map of simplicial sets

$$(d_0)_\bullet : s_\bullet S_2\mathscr{C} \to s_\bullet\mathscr{C}$$

satisfying the hypotheses of lemma B.

Proof of Proposition assuming the three lemmas.

Note that $s_0\mathscr{C} = *$ so by the three lemmas we have a homotopy fiber sequence

$$f/_{(0,*)} \to s_\bullet S_2\mathscr{C} \xrightarrow{(d_0)_\bullet} s_\bullet\mathscr{C}$$

and by inspection

$$f/_{(0,*)} = s_\bullet S_2'\mathscr{C} \quad \text{where } S_2'\mathscr{C} \subseteq S_2\mathscr{C}$$

is the full subcategory on objects of the form $0 \to B \xrightarrow{\cong} B$. Thus, there is a homotopy equivalence

$$j : s_\bullet\mathscr{C} \to f/_{(0,*)}.$$

we therefore have a homotopy equivalence

$$\begin{tikzcd} f/_{(0,-)} \arrow[r] & s_\bullet S_2\mathscr{C} \arrow[r] & s_\bullet \mathscr{C} \\ s_\bullet\mathscr{C} \arrow[r] \arrow[u, "\wr|", "j"'] & s_\bullet S_2\mathscr{C} \arrow[r] \arrow[u, "id"] & s_\bullet\mathscr{C} \arrow[u, "id"] \end{tikzcd}$$

and a map of fiber sequences

$$\begin{tikzcd} s_\bullet\mathscr{C} \arrow[r] \arrow[d, equal] & s_\bullet S_2\mathscr{C} \arrow[r] \arrow[d, shift right] & s_\bullet\mathscr{C} \arrow[d, equal] \\ s_\bullet\mathscr{C} \arrow[r] & s_\bullet\mathscr{C}\times s_\bullet\mathscr{C} \arrow[r] \arrow[u, shift right, "(v)_\bullet"'] & s_\bullet\mathscr{C} \end{tikzcd}$$

where

$v : \mathscr{C}\times\mathscr{C} \to S_2\mathscr{C}$ is the exact functor sending

$$(A,B) \longmapsto (A \rightarrowtail A \vee B \twoheadrightarrow B)$$

Consequently,

$$(v)_\bullet : |s_\bullet S_2\mathscr{C}| \simeq |s_\bullet\mathscr{C}|\times|s_\bullet\mathscr{C}|$$

is a homotopy equivalence. Since

$$((d_0)_\bullet,(d_1)_\bullet)\circ(v)_\bullet = id_{|s_\bullet\mathscr{C}|\times|s_\bullet\mathscr{C}|}$$ This implies

$((d_0)_\bullet,(d_1)_\bullet)$ is also a homotopy equivalence.

Finally, we prove the technical lemma.

Proof of technical lemma

For every $y \in S_n \mathcal{C}$ and every map $w : [n] \to [m]$ in Δ, we need to show that the induced map

$$w_* : f/_{(m, w^* y)} \to f/_{(n, y)}$$

is a homotopy equivalence. Any such w can be embedded in a triangle

$$\begin{tikzcd} {[n]} \arrow[r, "w"] & {[m]} \\ {[0]} \arrow[u, "u"] \arrow[ur, "v"'] & \end{tikzcd}$$

So it suffices to prove the result for maps of the form

$$v_i : [0] \to [n]$$

sending 0 to i.

Let x be the unique 0-simplex of $S_\bullet \mathcal{C}$

It suffices to show the induced map

$$(v_i)_\gamma : f/_{(0,\gamma)} \longrightarrow f/_{(n,y')}$$

is a homotopy equivalence for each $y' \in S_n \mathcal{C}$.

To do this, we define a left inverse

$$p : f/_{(n,y')} \longrightarrow S_\cdot \mathcal{C}$$

to the composite

$$S_\cdot \mathcal{C} \xrightarrow[\simeq]{j} f/_{(0,\gamma)} \xrightarrow{(v_i)_\gamma} f/_{(n,y')}$$

So that $p \circ (j \circ (v_i)_\gamma) = id_{S_\cdot \mathcal{C}}$. We will then show that

$$(j \circ (v_i)_\gamma) \circ p \simeq id_{f/_{(n,y')}},$$ which

implies that $(v_i)_\gamma$ is a homotopy equivalence.

Note that

$$S_n S_2 \mathcal{C} = \mathrm{ob}\, S_2(S_m \mathcal{C})$$

$$\Downarrow$$

$$A' \rightarrowtail A \twoheadrightarrow A'' \quad \text{s.t.} \quad A, A', A'' : \mathrm{Arr}([m]) \to \mathcal{C}$$

and for all $\theta : [1] \to [m]$

$$A'(\theta) \rightarrowtail A(\theta) \twoheadrightarrow A''(\theta).$$

Therefore, an n-simplex of $f/_{(n,\psi')}$

is an m-simplex

$$A' \rightarrowtail A \twoheadrightarrow A'' \in s_m S_2 \mathcal{C} = \mathrm{ob}\, S_2 S_m \mathcal{C}$$

and a map $\theta : [m] \to [n]$

s.t. A' is the composite

$$A' : \mathrm{Arr}([m]) \xrightarrow{\mathrm{Arr}(\theta)} \mathrm{Arr}([n]) \xrightarrow{\psi'} \mathcal{C} \quad \text{✱}$$

and

$$(d_2)_* : s_m S_2 \mathcal{C} = \mathrm{ob}\, S_2(S_m \mathcal{C}) \to s_m \mathcal{C} = \mathrm{ob}\, S_m \mathcal{C}$$

satisfies

$$(A' \rightarrowtail A \twoheadrightarrow A'') \longmapsto A''.$$

This induces a map

$$\begin{tikzcd}
p : f/_{(n,\psi')} \arrow[r, hook] & s_m S_2 \mathcal{C} \arrow[r, "(d_0)_*"] & s_m \mathcal{C} \arrow[dl, bend left] \\
& f/_{(0,*)} \arrow[ul, bend left] &
\end{tikzcd}$$

so

$$\begin{tikzcd}
0 \to A'' \xrightarrow{id} A'' \arrow[rr, mapsto] & & A'' \arrow[dl, mapsto, bend left] \\
& 0 \to A'' \xrightarrow{id} A'' \arrow[ul, mapsto, bend left] &
\end{tikzcd}$$

$$((v_i)_* \circ j) \circ p = \mathrm{id}_{s_m \mathcal{C}}$$

We just need to show

$$p \circ |v_!|_0 \circ j \simeq \mathrm{id}_{f/(n,y')}.$$

First, we give an explicit simplicial homotopy equivalence

$$\Delta^n \rightleftarrows \Delta^0$$

where $\Delta^n \to \Delta^0$ contracts Δ^n to its last vertex.

This is given by a natural transformation of functors

$$H : \Delta_{/[1]}^{op} \longrightarrow \Delta^{op} \xrightarrow{\Delta^n} \mathrm{Set}$$
$$([m] \to [1]) \longmapsto [m] \longmapsto \mathrm{Hom}_\Delta([m],[n])$$

to itself:

$$\Delta_{/[1]}^{op} \to \mathrm{Nat}(\Delta^n, \Delta^n)$$

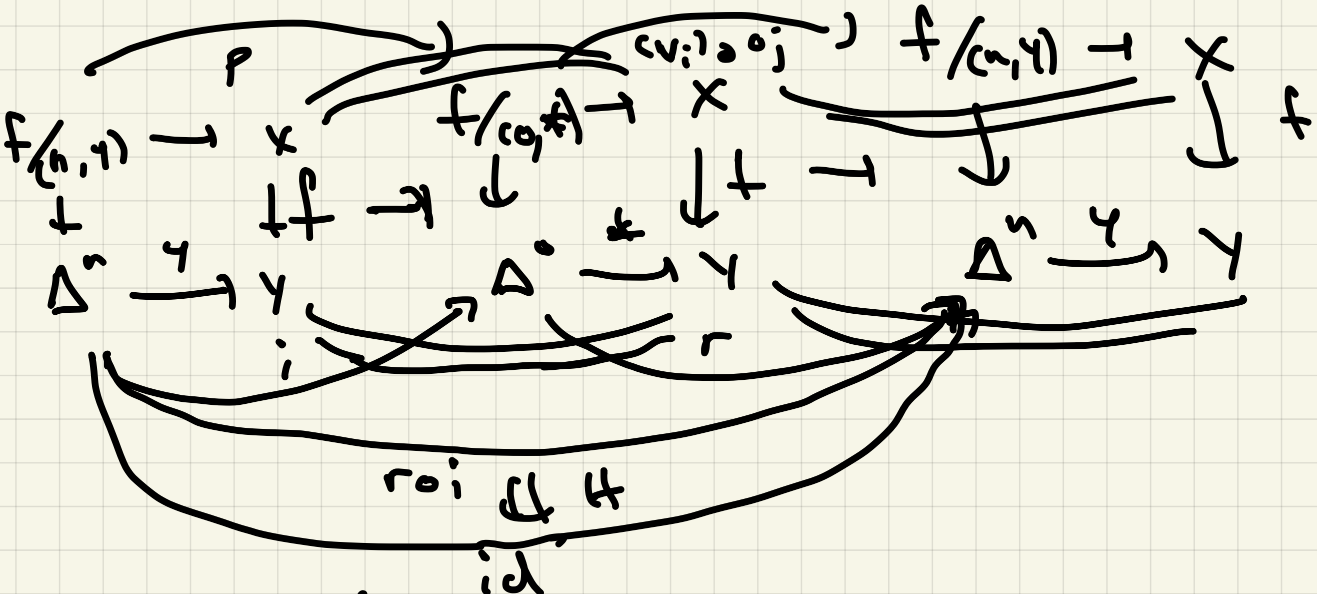
$$v : [n] \to [1] \longmapsto \Big((u : [m] \to [n]) \longmapsto (\bar{u} : [m] \to [n]) \Big)$$

where $\bar{u}$ is defined as the composite

$$\bar{u} : [m] \xrightarrow{(u,v)} [n] \times [1] \xrightarrow{w} [n]$$

where $w(j,0) = j \quad \forall\ 0 \leq j \leq n.$

$w(j,1) = n$



$$
\begin{tikzcd}
f/(n,y) \arrow[r, bend left, "p\circ(v_i)_*\circ j"] \arrow[r, bend right, "\mathrm{id}_{f/(n,y)}"'] \arrow[r, phantom, "\Downarrow H'"] \arrow[d] & f/(n,y) \arrow[d] \\
\Delta^n \arrow[r, bend left, "i\circ r"] \arrow[r, bend right, "\mathrm{id}_{\Delta^n}"'] \arrow[r, phantom, "\Downarrow H"] & \Delta^n
\end{tikzcd}
$$

$$
\begin{tikzcd}
f/(n,y)\times\Delta^1 \arrow[r, dashed, "H'"'] \arrow[d] & f/(n,y) \arrow[d] \\
\Delta^n\times\Delta^1 \arrow[r] & \Delta^n
\end{tikzcd}
$$

In other words, we will show that we can lift this homotopy to a homotopy $p\circ(v_i)_*\circ j \simeq \mathrm{id}_{f/(n,y)}$

Such a homotopy

$$H' : \Delta/[1]^{op} \longrightarrow \mathrm{Nat}\big(f/_{(n,y')}, f/_{(n,y')}\big)$$

sends

$$(v : [m] \to [1]) \longmapsto \begin{array}{c} \big(A' \rightarrowtail A \to A'',\ u : [m] \to [n] \\ (A' : \mathrm{Arr}([m]) \to \mathrm{Arr}([n]) \xrightarrow{y'} \mathscr{C})\big) \\ \big\downarrow \\ \big(\bar{A}' \rightarrowtail \bar{A} \twoheadrightarrow \bar{A}'' : \bar{u} : [m] \to [n] \\ (\bar{A}' : \mathrm{Arr}([m]) \xrightarrow{\mathrm{Arr}(\bar{u})} \mathrm{Arr}([n]) \to \mathscr{C})\big) \end{array}$$

where $\bar{u}$ is defined as before.

We therefor need to say how to construct $\bar{A}' \to \bar{A} \to \bar{A}''$ from $A' \rightarrowtail A \to A''$ and $\bar{u}$.

Since

$$\bar{A}' : \mathrm{Arr}([m]) \xrightarrow{\mathrm{Arr}(\bar{u})} \mathrm{Arr}([n]) \longrightarrow \mathscr{C}$$

this part is forced. We therefore define $\bar{A}$ as the pushout

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d] & A \arrow[d] \\ \bar{A}' \arrow[r, tail] & \bar{A} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

and $\bar{A}''$ as the pushout

$$\begin{tikzcd} \bar{A}' \arrow[r, tail] \arrow[d] & \bar{A} \arrow[d] \\ 0 \arrow[r, tail] & \bar{A}'' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

To make this compatible with all of the other structure, we need explicit choices of pushout such that

1) $\bar{A}$ is the <u>object-wise pushout</u>,

(This implies that after applying face and degeneracy maps

$S_n\mathcal{C} \to S_m\mathcal{C}$

we still have a pushout.)

2) if $A' \to \bar{A}'$ is $\mathrm{id}_{A'}$, we insist that $A \to \bar{A}$ is id_A,

3) if $\bar{A} = 0$, we insist that $A \to \bar{A}$ is the map $A \to A''$ so that

$$\begin{tikzcd}
\bar{A}' \arrow[r, tail] \arrow[d, equal] & \bar{A} \arrow[r, two heads] \arrow[d, equal] & \bar{A}'' \arrow[d, equal] \\
0 \arrow[r, tail] & A'' \arrow[r, "\mathrm{id}"] & A''
\end{tikzcd}$$

By building these choices into the definition of $A' \rightarrowtail \bar{A} \twoheadrightarrow \bar{A}''$, all compatibility holds. $\square$