Lecture 14: Localization

I. Motivation

Let R be a Noetherian ring. Let S be a central multiplicatively closed set in R. Define

$$M_S(R) \subseteq M(R)$$

to be the full subcategory of finitely generated S-torsion R-modules. Then $M(R)$ is an abelian category and $M_S(R)$ is an abelian subcategory that is closed under subobjects, quotients, and extensions.

<u>Def</u>. A Serre subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a subabelian category that is closed under subobjects, quotients, and extensions.

We can then form a quotient category $M(R)/M_S(R)$ and identify it with $M(S^{-1}R)$. More generally given a serre subcategory $\mathcal{B} \subseteq \mathcal{A}$ we can form a quotient category $\mathcal{A}/\mathcal{B}$ using Gabriel's calculus of fractions.

__Construction__ We define $\mathcal{A}/\mathcal{B}$ to be a category with the same objects as $\mathcal{A}$. We say a map $f: A \to A'$ in $\mathcal{A}$ is a $\mathcal{B}$-isomorphism if $\ker f$ and $\operatorname{coker} f$ are objects in $\mathcal{B}$. The morphisms in $\mathcal{A}/\mathcal{B}$ are equivalence classes of spans

$$\begin{tikzcd} & A' \arrow[dl] \arrow[dr] & \\ A_1 & & A_2 \end{tikzcd}$$

where $A' \to A'$ is a $\mathcal{B}$-isomorphism.

Two spans are in the same equivalence class if there is a commuting diagram

$$\begin{tikzcd} & A' \arrow[dl] \arrow[dr] & \\ A_1 & A \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_2 \\ & A'' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where both of $A \to A'$ and $A \to A''$ are $\mathcal{B}$-isomorphisms.

Composition of two spans

$$\begin{tikzcd}[column sep=small] & A' \arrow[dl] \arrow[dr] & \\ A_1 & & A_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & A'' \arrow[dl] \arrow[dr] & \\ A_2 & & A_2 \end{tikzcd}$$

is defined by the pullback

$$\begin{tikzcd}[column sep=small] & & A \arrow[dl] \arrow[dr] & & \\ & A' \arrow[dl] \arrow[dr] & & A'' \arrow[dl] \arrow[dr] & \\ A_1 & & A_2 & & A_3 \end{tikzcd}.$$

There is an evident functor

$$\mathrm{loc}: \mathcal{A} \to \mathcal{A}/\mathcal{A}/\mathcal{B}$$

$$A \longmapsto A$$

$$A' \to A_2 \quad \longmapsto \quad \begin{tikzcd}[column sep=small] & A' \arrow[dl, equal] \arrow[dr] & \\ A' & & A_2 \end{tikzcd}$$

<u>Prop</u>. The construction $\mathcal{A}/\mathcal{B}$ is an abelian category and the functor

$$\mathrm{loc}: \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

is exact.

<u>Pf</u>. See Swan "Algebraic K-theory" p. 44 ff, for example.

Rem. If $B \in \mathcal{B}$ then

$B \in \mathcal{A}/\mathcal{B}$ is isomorphic to 0

since
$$\begin{tikzcd}[column sep=small]
 & & B \arrow[dl, equal] \arrow[dr, equal] & & \\
 & B \arrow[dl] \arrow[dr, equal] & & B \arrow[dl, equal] \arrow[dr] & \\
0 & & B & & 0
\end{tikzcd}$$
is the unique map $0 \to 0$

and
$$\left[\begin{tikzcd}[column sep=small]
 & & B\times B \arrow[dl] \arrow[dr] & & \\
 & B \arrow[dl, equal] \arrow[dr] & & B \arrow[dl] \arrow[dr, equal] & \\
B & & 0 & & B
\end{tikzcd}\right] = \left[\begin{tikzcd}[column sep=small]
 & B \arrow[dl, equal] \arrow[dr, equal] & \\
B & & B
\end{tikzcd}\right]$$

by
$$\begin{tikzcd}
 & B\times B \arrow[dl] \arrow[dr] \arrow[d, equal] & \\
B & B\times B \arrow[l] \arrow[r] & B \\
 & B \arrow[u] \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

Similarly, maps in $\mathcal{B}$ are sent to isomorphisms in $\mathcal{A}/\mathcal{B}$, which are exactly maps
$$\begin{tikzcd}[column sep=small]
 & A' \arrow[dl] \arrow[dr] & \\
A_1 & & A_2
\end{tikzcd}$$
where both legs are $\mathcal{B}$-isos

<u>Thm</u>. (Localization)

Let B be a Serre subcategory of a small abelian category A. Then there is a homotopy fiber sequence

$$K^Q(B) \to K^Q(A) \to K^Q(A/B)$$

inducing a LES in homotopy groups

$$\begin{aligned} &\to K_i^Q(B) \to K_i^Q(A) \to K_i^Q(A/B) \to \\ &\to K_{i-1}^Q(B) \to K_{i-1}^Q(A) \to K_{i-1}^Q(A/B) \to \cdots \end{aligned}$$

<u>Cor</u>: Let R be a Noetherian ring and S a central multiplicatively closed subset. Then there is a homotopy fiber sequence

$$K^Q(M_S(R)) \to G(R) \to G(S^{-1}R)$$

Rem. The construction $\mathcal{A}/\mathcal{B}$ has a universal property that functors $\mathcal{A} \xrightarrow{F} \mathcal{C}$ where $\mathcal{C}$ is an abelian category and

$$F(B) \cong 0 \quad \text{for all } B \in \mathcal{B}$$

and $F(g)$ is an isomorphism in $\mathcal{C}$ for all maps g in $\mathcal{B}$ factor as

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr] & \mathcal{A}/\mathcal{B} \arrow[d] \\ & \mathcal{C} \end{tikzcd}.$$

In particular, there factorization

$$\begin{tikzcd} \mathcal{M}(R) \arrow[r] \arrow[dr] & \mathcal{M}(R)/\mathcal{M}_S(R) \arrow[d, dashed] \\ & \mathcal{M}(S^{-1}R) \end{tikzcd}$$

and it is easy to check that this functor is an equivalence of categories.

<u>Cor</u>: Let R be a Dedekind domain with field of fractions F. Then there is a homotopy fiber sequence

$$\prod_{\substack{p \subset R \\ \text{prime} \\ \text{ideal}}} K(R/p) \to K(R) \to K(F).$$

<u>Pf</u>. By the previous corollary, there is a fiber sequence

$$K^Q(M_{R-\{0\}}(R)) \to G(R) \to G(F).$$

By the Resolution theorem

$$G(R) \simeq K(R) \quad \text{and} \quad G(F) \simeq K(F).$$

By Dévissage and Resolution

$$K^Q(M_{R-\{0\}}(R)) \simeq \prod_{p \subset R} G(R/p)$$

(See example from last lecture.) $\simeq \prod_{p \subset R} K(R/p)$.

Proof of Localization

We will apply Quiller's theorem B, so we need to show

1) $BQB \simeq B({}_0\backslash Q\mathrm{loc})$
   where $Q\mathrm{loc} : QA \to Q(A/B)$.

and

2) $B({}_L\backslash Q\mathrm{loc}) \xrightarrow{\simeq} B({}_{L'}\backslash Q\mathrm{loc})$

   for every map $L \to L'$ in QA/B

We break this down further into several steps. First, we consider full subcategories $F_L \subseteq {}_L\backslash Q\mathrm{loc}$.

The steps will then be to show the following

Step 0. $BQB \simeq BF_0$

Step 1. For all L in QA/B,

$$BF_L \simeq B_{L\backslash}Q^{loc}.$$

Step 2. The categories F_L can be written as a filtered colimit

$$\operatorname*{colim}_{N \in I_L} \mathcal{E}_N$$

of categories $\mathcal{E}_N$

Step 3. Each $\mathcal{E}_N$ has a full subcategory $\mathcal{E}_N' \subseteq \mathcal{E}_N$ such that

$$B\mathcal{E}_N' \simeq BQB.$$

Step 4. The inclusion induces a homotopy equivalence $B\mathcal{E}_N' \simeq B\mathcal{E}_N$.

we will do this in such a way that the homotopy equivalence

$$BQB \simeq B(\operatorname{colim}_{N \in I_L} \Sigma_N) \simeq BF_L$$

is compatible with the maps

$$\begin{tikzcd} BF_L \arrow[r] & BF_{L'} \\ BQB \arrow[u, "\wr"] \arrow[ur, "\wr"'] & \end{tikzcd}$$

We write $\bar{A} := loc(A)$ and $\bar{h} := loc(h)$.

We start by defining $F_L \subseteq \Delta Q loc$ to be the full subcategory consisting of objects $(A, \overset{\in A}{v} : L \to \bar{A} \in QA/B)$

where $L \twoheadleftarrow L' \xrightarrow{B\text{-iso}} W \xleftarrow{B\text{-iso}} \bar{A}' \to \bar{A}$

has the property that all maps are B-isomorphisms.

Then when $L = 0$, $F_0 \cong QB$

<u>Step 1</u> We use Quillen's theorem A to prove $i_L : F_L \hookrightarrow {}_L\backslash Qloc$ induces a homotopy equivalence $Bi_L : BF_L \to B_L\backslash Qloc$.

We need to show $Bi_L/_{(A,u)} \simeq *$ for all pairs (A,u) in ${}_L\backslash Qloc$. Note that a map in $Q\mathcal{C}$

$$\begin{tikzcd} & B_1 \arrow[dl, two heads] \arrow[dr, tail] & \\ A & & B \end{tikzcd}$$

is the same data as a sequence

$$B_2 \rightarrowtail B_1 \rightarrowtail B, \text{ denoted } (B_2, B_1)$$

and if we fix B, then these have an ordering by

$(B_2, B_1) \leq (B_3, B_4)$ if there is a

sequence $B_4 \rightarrowtail B_2 \rightarrowtail B_1 \rightarrowtail B_3 \rightarrowtail B$

of monomorphisms.

Consequently, for example $i_L/(A,U)$ forms a partially ordered set and it is even filtered since given (A_1, A_2) and (A_3, A_4) there exists $(A_1 \cap A_3, A_2 + A_4)$ such that

$$(A_1, A_2) \leq (A_1 \cap A_3, A_2 + A_4) \geq (A_3, A_4).$$

Since filtered categories are contractible,

$$B\, i_L/(A,U) \simeq * \quad \text{and}$$

$$BF_L \simeq B_L \backslash Q\mathrm{loc}.$$

Note that this is compatible with morphisms $L \to L'$ in QA/B.

Step 2. We define categories Σ_N for N in $\mathcal{A}$ to be the category of pairs $(A, h : A \to N \in \mathcal{A})$ where $\bar{h}$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. A morphism between pairs is a span

$$A \twoheadleftarrow A'' \xrightarrow{\sim} A'$$

in $Q\mathcal{A}$ such that the two composites $A'' \to N$ agree.

We define

$$k_N : \Sigma_N \to Q\mathcal{B}$$

by

$$(A, h) \longmapsto \ker h$$

We define $\Sigma_N' \subseteq \Sigma_N$ to be the full subcategory on pairs (A, h) such that h is an epimorphism. Write

$$k_N' : \Sigma_N' \hookrightarrow \Sigma_N \to Q\mathcal{B}.$$

We further define a category

I_L where $L \in A/B$ with

objects $(N, \alpha : \overline{N} \xrightarrow{\cong} L)$ where N

is an object in A and $\overline{N} \xrightarrow{\cong} L$ is an

isomorphism in A/B. The morphisms

$(N,\alpha) \to (N',\beta)$ in I_L are B-isomorphisms

$g : N \to N'$ in A such that

$$\begin{tikzcd} \overline{N} \arrow[rr, "\overline{g}"] \arrow[dr, "\alpha"'] & & \overline{N}' \arrow[dl, "\beta"] \\ & L & \end{tikzcd}$$ commutes

This is a filtered category. We just check one of the properties to illustrate this:

Given $g_1, g_2 : (N, \alpha) \longrightarrow (N', \beta)$

then $\overline{g}_1 - \overline{g}_2 = 0$ so $\mathrm{im}(g_1 - g_2) \in B$

Let $N'' = N' / \mathrm{im}(g_1 - g_2)$ and
$$\begin{tikzcd} \overline{N}' \arrow[r, "\cong"] \arrow[dr, "\beta"'] & \overline{N' / \mathrm{im}(g_1 - g_2)} \arrow[d, dashed, "\gamma"] \\ & L \end{tikzcd}$$

Then

$(N, \alpha) \rightrightarrows (N', \beta) \longrightarrow (N'', \gamma)$ is a coequalizer.

We define a functor

$$I_L \longrightarrow \mathrm{Cat} \quad (\text{category of small categories})$$

$$(N,\alpha) \longmapsto \Sigma_N$$

$$(N,\alpha) \to (N',\beta) \longmapsto \Sigma_N \to \Sigma_{N'}$$

$g: N \to N'$ $\quad (N, h: A \to N) \longmapsto (N', g \circ h)$

β-iso

We also define functors

$$P_{(N,\alpha)}: \Sigma_N \longrightarrow F_L$$

for fixed L and arbitrary $(N,\alpha) \in I_N$

by

$$(N, h: M \to N) \longmapsto (M, \bar{h}^{-1} \circ \alpha^{-1}: L \xrightarrow{\cong} \bar{N} \xrightarrow{\cong} \bar{M})$$

and for maps $g: (N,\alpha) \to (N,\beta)$

$$P_{(N',\beta)} \circ g_* = P_{(N,\alpha)}.$$

So there is a map

$$\operatorname{colim}_{(N,\alpha) \in I_N} \Sigma_N \longrightarrow F_L.$$

<u>Step 2</u> The map

$$\operatorname*{colim}_{(N,\alpha)\in I_N} \Sigma_N \to F_L$$

is a isomorphism of categories.

We just check on objects because the argument on morphisms is basically the same.

Note that $(M, \theta : L \xrightarrow{\cong} \overline{M}) = P_{(M,\theta^{-1})}(M, id_M)$

for any (M,θ) in F_L. So this map is surjective on objects. Given

$$P_{(N,\alpha)}(N,h) = P_{(N,\alpha)}(N',h')$$

then $N = N'$ and $\overline{h} = \overline{h'}$. Letting

$N' = \text{im}/(h-h')$ we produce map

$(N,\alpha) \to (N',\beta)$ s.t. $g_*(N,h) = g_*(N,h')$

So as objects in $\operatorname*{colim}_{(N,\alpha)\in I_N} \Sigma_N$ $(N,h) = (N',h')$.

Step 3. We already constructed

$\mathcal{E}_N^!$ and functors $k_N^! : \mathcal{E}_N^! \to Q\mathcal{B}$.

We again apply Quillen's theorem A.

It suffices to show $B(k_N^!/_B) \simeq *$

for all objects $B \in \mathcal{B}$. Note that

$k_N^!/_B$ is fibered over $\mathcal{E}_N^!$

and it has objects $((m,h), u)$

where $(m,h) \in \mathcal{E}_N^!$ and $u : B \to \ker h$

is a map in $Q\mathcal{B}$. Let $\mathcal{C}$ be

the full subcategory of $k_N^!/_B$ consisting

of $((m,h), u)$ such that u is of

the form $j^! : B \overset{/\!/B}{\twoheadleftarrow} \ker h$.

Since every map u in $Q\mathcal{B}$ is of

the form $u = j^! i_!$ for some epi j and

mono i,

Given $((M,h),v) \in \ker h/_{B}$

We write

$$
v : \begin{tikzcd}[column sep=small]
 & W \arrow[dl] \arrow[dr] & \\
\ker h & & B
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & W \arrow[dl] \arrow[dr] & & \\
 & \ker h \arrow[dl, equal] \arrow[dr, dashed, "i"'] & & B \arrow[dl] \arrow[dr, equal] & \\
\ker h & & B_0 & & B
\end{tikzcd}
$$

So $v = j^! \circ i_!$

and define

$$
\begin{tikzcd}
0 \arrow[r] & \ker h \arrow[r, hook] \arrow[d, "v", "i"'] \arrow[dr, phantom, "\ulcorner"] & M \arrow[r, two heads, "h"] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_0 \arrow[r] & i_* M \arrow[r, "\tilde{h}"] & N \arrow[r] & 0
\end{tikzcd}
$$

Then $((M,h),v) \xmapsto{i_*} ((i_*M, \tilde{h}), j^!)$

is adjoint to the inclusion

$$\mathcal{C} \rightleftarrows \ker h/_{B}.$$

$j_B : B \twoheadrightarrow 0$

Moreover, $((N, id_N), j_B^!)$

is initial in $\mathcal{C}$ so $B\mathcal{C} \simeq *$.

This is homotopy equivalence

is compatible w/ $g_r|_{\mathcal{E}'_N}$

$$\begin{tikzcd} B\mathcal{E}'_N \arrow[dr, "\simeq"] \arrow[d, "\simeq"] & \\ B\mathcal{E}_{N'}' \arrow[r, "\simeq"'] & BQB \end{tikzcd}$$

$Bg_\otimes$

We now prove $\mathcal{E}'_N \hookrightarrow \mathcal{E}_N$

induces a homotopy equivalence $B\mathcal{E}'_N \simeq B\mathcal{E}_N$

Let $\mathcal{I}$ be the ordered set of subobjects

$I \rightarrowtail N$ of N s.t. $N/I \in \mathcal{B}$.

regarded as a category. We observe

that there is a functor

$$p_N : \mathcal{E}_N \longrightarrow \mathcal{I}$$
$$(m, n) \longmapsto \operatorname{im} h$$

which is fibered with fiber $p_N^{-1}(I) = \mathcal{E}'_I$.

The base change functor

$$p_N^{-1}(I \to J) : \underset{\substack{\shortparallel \\ \Sigma_I^1}}{p_N^{-1}(I)} \to \underset{\substack{\shortparallel \\ \Sigma_J^1}}{p_N^{-1}(J)}$$

$$(M, M \twoheadrightarrow I) \longmapsto (J \times_I M, J \times_I M \twoheadrightarrow J)$$

This clearly commutes with the inclusions

$$\begin{tikzcd} \Sigma_I^1 \arrow[r] \arrow[dr, hook] & \Sigma_J^1 \arrow[d, hook] \\ & \Sigma_N \end{tikzcd}.$$

By the previous result

$$\begin{tikzcd} B\Sigma_I^1 \arrow[r] \arrow[dr, "\simeq"'] & B\Sigma_J^1 \arrow[d, "\simeq"] \\ & BQB \end{tikzcd}$$

So we can apply Quillen's theorem B.

Also, $BI \simeq 0$ since I is a filtered partially ordered set. So $B\Sigma_N^1 \xrightarrow{\simeq} B\Sigma_N$.

Note that these are also compatible with the maps

$$\begin{tikzcd}
P_{(N,\alpha)} : \Sigma_N \arrow[r] \arrow[d] & F_L \arrow[d] \\
P_{(M,\beta)} : \Sigma_M \arrow[r] & F_{L'}
\end{tikzcd}$$

So

$$\begin{tikzcd}
QB \simeq \operatorname{B colim}_{I_L} \Sigma_N \arrow[r, "\simeq"] \arrow[d, phantom, "\wr"] & BF_L \arrow[r, "\simeq"] \arrow[d, "\wr"] & B_L\backslash Q\mathrm{Loc} \arrow[d, "\wr"] \\
QB \simeq \operatorname{B colim}_{I_{L'}} \Sigma_M \arrow[r, "\simeq"] & BF_{L'} \arrow[r, "\simeq"] & B_{L'}\backslash Q\mathrm{loc}
\end{tikzcd}$$

.

$\square$

By Quillen,

$$K_i(\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/(p^k-1) & i=2k-1>0 \\ 0 & i=2k \end{cases} \qquad p \text{ prime}$$

So if you know $K_i(\mathbb{Q})$, then you can compute $K_i(\mathbb{Z})$ by the LES

$$\to \prod_p K_i(\mathbb{Z}/p) \to K_i(\mathbb{Z}) \to K_i(\mathbb{Q}) \to$$

$$\to \prod_p K_{i-1}(\mathbb{Z}/p) \to K_{i-1}(\mathbb{Z}) \to K_{i-1}(\mathbb{Q}) \to$$

$$\to \cdots$$

There is an injection

Ex: $K_{2i}(\mathbb{Z}) \hookrightarrow K_{2i}(\mathbb{Q})$

$i \geq 0$