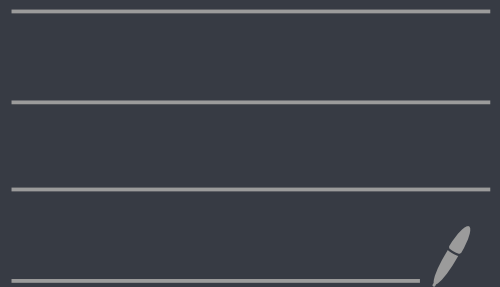


# Lecture 14: Localization

---



## I. Motivation

Let  $R$  be a Noetherian ring. Let  $S$  be a central multiplicatively closed set in  $R$ . Define

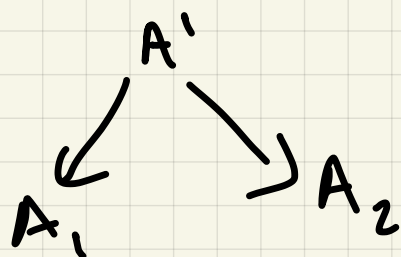
$$M_S(R) \subseteq M(R)$$

to be the full subcategory of finitely generated  $S$ -torsion  $R$ -modules. Then  $M(R)$  is an abelian category and  $M_S(R)$  is an abelian subcategory that is closed under subobjects, quotients, and extensions.

Def. A **Serre subcategory**  $\mathcal{B}$  of an abelian category  $\mathcal{A}$  is a subabelian category that is closed under subobjects, quotients, and extensions.

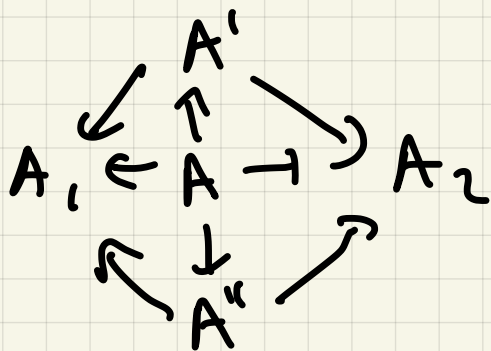
We can then form a quotient category  $M(R)/M_S(R)$  and identify it with  $M(S^{-1}R)$ . More generally given a Serre subcategory  $\mathcal{B} \subseteq \mathcal{A}$  we can form a quotient category  $\mathcal{A}/\mathcal{B}$  using Gabriel's calculus of fractions.

Construction We define  $A/B$  to be a category with the same objects as  $A$ . We say a map  $f: A \rightarrow A'$  in  $A$  is a  $\mathcal{B}$ -isomorphism if  $\ker f$  and  $\operatorname{coker} f$  are objects in  $\mathcal{B}$ . The morphisms in  $A/B$  are equivalence classes of spans



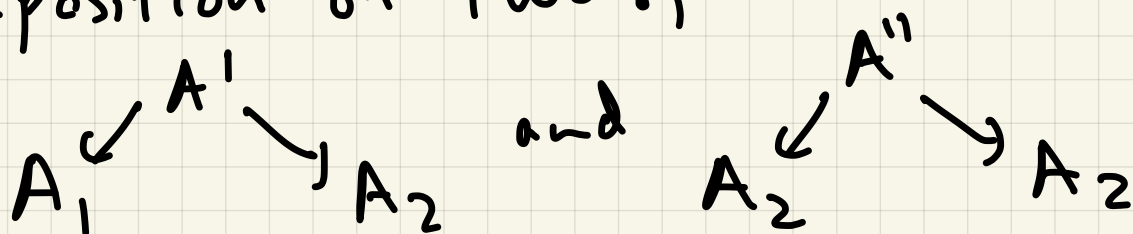
where  $A' \rightarrow A'$  is a  $\mathcal{B}$ -isomorphism.

Two spans are in the same equivalence class if there is a commuting diagram

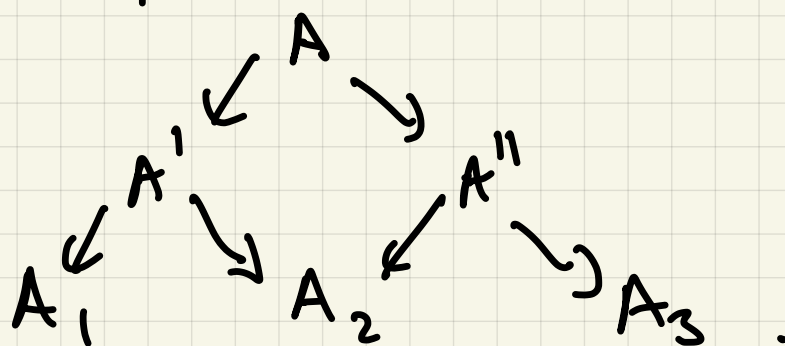


where both of  $A \rightarrow A'$  and  $A \rightarrow A''$  are  $\mathcal{B}$ -isomorphisms.

Composition of two spans



is defined by the pullback



There is an evident functor

$$\text{loc}: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}/\mathcal{B}$$

$$\begin{array}{ccc}
 A \longmapsto A \\
 A' \rightarrow A_2 \longmapsto A' \begin{array}{l} \longleftarrow A' \\ \longrightarrow A_2 \end{array}
 \end{array}$$

Prop. The construction  $\mathcal{A}/\mathcal{B}$  is an abelian category and the functor

$$\text{loc}: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$$

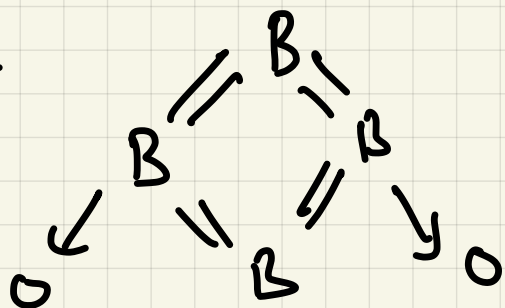
is exact.

Pf. See Swan "Algebraic K-theory" p. 44 ff, for example.

Rem. If  $B \in \mathcal{B}$  then

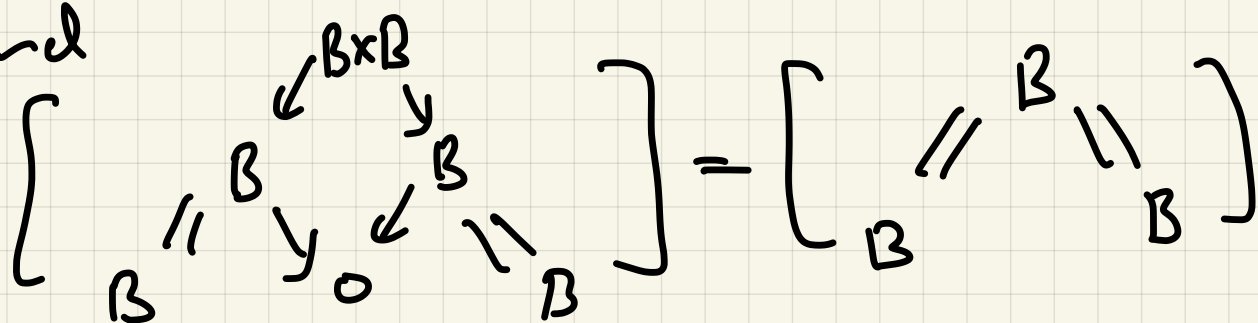
$B \in \mathcal{A}/\mathcal{B}$  is isomorphic to 0

since

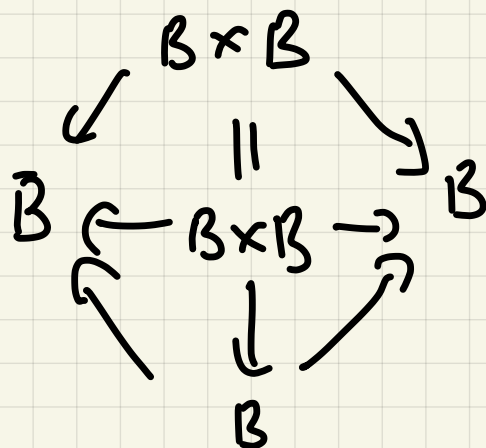


is the unique map  $0 \rightarrow 0$

and



by



Similarly, maps in  $\mathcal{B}$  are sent to

isomorphisms in  $\mathcal{A}/\mathcal{B}$ , which are

exactly maps  $A_1 \xrightarrow{A'} A_2$  where both legs are  $\mathcal{B}$ -isos

## Thm. (Localization)

Let  $\mathcal{B}$  be a Serre subcategory of a small abelian category  $\mathcal{A}$ . Then there is a homotopy fiber sequence

$$K^{\mathcal{Q}}(\mathcal{B}) \rightarrow K^{\mathcal{Q}}(\mathcal{A}) \rightarrow K^{\mathcal{Q}}(\mathcal{A}/\mathcal{B})$$

inducing a LES of homotopy groups

$$\rightarrow K_i^{\mathcal{Q}}(\mathcal{B}) \rightarrow K_i^{\mathcal{Q}}(\mathcal{A}) \rightarrow K_i^{\mathcal{Q}}(\mathcal{A}/\mathcal{B}) \rightarrow$$

$$\rightarrow K_{i-1}^{\mathcal{Q}}(\mathcal{B}) \rightarrow K_{i-1}^{\mathcal{Q}}(\mathcal{A}) \rightarrow K_{i-1}^{\mathcal{Q}}(\mathcal{A}/\mathcal{B}) \rightarrow \dots$$

Cor: Let  $R$  be a Noetherian ring and  $S$  a central multiplicatively closed subset. Then

there is a homotopy fiber sequence

$$K^{\mathcal{Q}}(M_S(R)) \rightarrow G(R) \rightarrow G(S^{-1}R)$$

Rem. The construction  $A/B$  has a universal property that factors  $A \xrightarrow{F} \mathcal{C}$  where  $\mathcal{C}$  is an abelian category and

$$F(B) \cong 0 \text{ for all } B \in \mathcal{B}$$

and  $F(g)$  is an isomorphism in  $\mathcal{C}$  for all maps  $g$  in  $\mathcal{B}$  factor as

$$\begin{array}{ccc} A & \longrightarrow & A/B \\ & \searrow & \downarrow \\ & & \mathcal{C} \end{array}$$

In particular, here factorization

$$\begin{array}{ccc} M(R) & \longrightarrow & M(R)/M_S(R) \\ & \searrow & \downarrow \\ & & M(S^{-1}R) \end{array}$$

and it is easy to check that this functor is an equivalence of categories.

Cor: Let  $R$  be a Dedekind domain with field of fractions  $F$ . Then there is a homotopy fiber sequence

$$\prod_{\substack{p \subset R \\ \text{prime} \\ \text{ideal}}} K(R/p) \rightarrow K(R) \rightarrow K(F).$$

Pf. By the previous corollary, there is a fiber sequence

$$K^{\mathbb{Q}}(M_{R\text{-tors}}(R)) \rightarrow G(R) \rightarrow G(F).$$

By the Resolution theorem

$$G(R) \simeq K(R) \quad \text{and} \quad G(F) \simeq K(F).$$

By Dévissage and Resolution

$$K^{\mathbb{Q}}(M_{R\text{-tors}}(R)) \simeq \prod_{p \subset R} G(R/p)$$

(See example from last lecture.)  $\simeq \prod_{p \subset R} K(R/p)$ .



# Proof of Localization

We will apply Quillen's theorem B,  
so we need to show

$$1) \quad BQB \simeq B(O \setminus Qloc)$$

where  $Qloc : QA \rightarrow Q(A/B)$ .

and

$$2) \quad B(L \setminus Qloc) \xrightarrow{\cong} B(L' \setminus Qloc)$$

for every map  $L \rightarrow L'$  in  $QA/B$

We break this down further into  
several steps. First, we consider  
full subcategories  $F_L \subseteq L \setminus Qloc$ .

The steps will then be to  
show the following

Step 0.  $BQB \simeq BF_0$

Step 1. For all  $L$  in  $QA/B$ ,

$$BF_L \simeq B_L \backslash Qloc.$$

Step 2. The categories  $F_L$  can be written as a filtered colimit

$$\operatorname{colim}_{N \in \mathbb{I}_L} \Sigma_N$$

of categories  $\Sigma_N$

Step 3. Each  $\Sigma_N$  has a full subcategory

$$\Sigma_N' \subseteq \Sigma_N \text{ such that}$$
$$B \Sigma_N' \simeq BQB.$$

Step 4. The inclusion induces a homotopy equivalence  $B \Sigma_N' \simeq B \Sigma_N$ .

We will do this in such a way that the homotopy equivalence

$$BQB \cong \underset{N \in I_L}{B(\operatorname{colim} \Sigma_N)} \cong BF_L$$

is compatible with the maps

$$BF_L \rightarrow BF_{L'}$$

$$\begin{array}{c} \downarrow \quad \swarrow \\ BQB \end{array}$$

We write  $\bar{A} := \operatorname{loc}(A)$  and  $\bar{h} := \operatorname{loc}(h)$ .

We start by defining  $F_L \subseteq \mathcal{L}Q/\operatorname{loc}$  to be the full subcategory consisting of objects  $(A, \nu: L \rightarrow \bar{A} \in \mathcal{Q}A/B)$

where

$$\begin{array}{ccc} & L' & \bar{A}' \\ & \swarrow & \searrow \\ L & \xrightarrow{B\text{-iso}} & \bar{A} \\ & \nwarrow & \swarrow \\ & W & \end{array}$$

has the property that all maps are B-isomorphisms.

Then when  $L=0$ ,  $F_0 \cong QB$

Step 1 We use Quillen's theorem A to

prove  $i_L: F_L \hookrightarrow L \setminus Qloc$  induces

a homotopy equivalence  $Bi_L: BF_L \rightarrow B(L \setminus Qloc)$ .

We need to show  $Bi_L / (A, \nu) \cong *$

for all pairs  $(A, \nu)$  in  $L \setminus Qloc$ . Note

that a map in  $Q\mathcal{C}$

$$\begin{array}{ccc} & B_1 & \\ \swarrow & & \searrow \\ A & & B \end{array}$$

is the same data as a sequence

$$B_2 \twoheadrightarrow B_1 \twoheadrightarrow B, \text{ denoted } (B_2, B_1)$$

and if we fix  $B$ , then these have an

ordering by

$$(B_2, B_1) \leq (B_0, B_3) \text{ if there is } \hookrightarrow$$

$$\text{Sequence } B_4 \twoheadrightarrow B_2 \twoheadrightarrow B_1 \twoheadrightarrow B_3 \twoheadrightarrow B$$

of monomorphisms.

Consequently, for example  $i_L / (A, U)$

forms a partially ordered set

and it is even filtered since given

$(A_1, A_2)$  and  $(A_3, A_4)$

there exists  $(A_1 \cap A_3, A_2 + A_4)$

such that

$$(A_1, A_2) \leq (A_1 \cap A_3, A_2 + A_4) \geq (A_3, A_4).$$

Since filtered categories are contractible,

$$B_{i_L / (A, U)} \simeq \mathbb{R} \text{ and}$$

$$BF_L \simeq B_L \setminus \text{Qloc}.$$

Note that this is compatible with

morphisms  $L \rightarrow L'$  in  $\mathcal{O}A/B$ .

Step 2. We define categories  $\Sigma_N$

for  $N$  in  $\mathcal{A}$  to be the category of pairs  $(A, h: A \rightarrow N \in \mathcal{A})$  where  $h$  is an isomorphism in  $\mathcal{A}/\mathcal{B}$ . A morphism between pairs is a square

$$A \leftarrow A'' \rightarrow A'$$

in  $\mathcal{Q}\mathcal{A}$  such that the two composites  $A'' \rightarrow N$  agree.

We define

$$k_N: \Sigma_N \rightarrow \mathcal{Q}\mathcal{B}$$

by  $(A, h) \mapsto \ker h$

We define  $\Sigma'_N \subseteq \Sigma_N$  to be the

full subcategory of pairs  $(A, h)$  such that  $h$  is an epimorphism. Write

$$k'_N: \Sigma'_N \hookrightarrow \Sigma_N \rightarrow \mathcal{Q}\mathcal{B}.$$

We further define a category

$\mathcal{I}_L$  where  $L \in \mathcal{A}/\mathcal{B}$  with objects  $(N, \alpha: \bar{N} \xrightarrow{\cong} L)$  where  $N$  is an object in  $\mathcal{A}$  and  $\bar{N} \xrightarrow{\cong} L$  is an isomorphism in  $\mathcal{A}/\mathcal{B}$ . The morphisms  $(N, \alpha) \rightarrow (N', \beta)$  in  $\mathcal{I}_L$  are  $\mathcal{B}$ -isomorphisms  $g: N \rightarrow N'$  in  $\mathcal{A}$  such that

$$\begin{array}{ccc} \bar{N} & \xrightarrow{\bar{g}} & \bar{N}' \\ \alpha \downarrow & & \downarrow \beta \\ L & & L \end{array} \text{ commutes}$$

This is a filtered category. We just check one of the properties to illustrate this:

Given  $g_1, g_2: (N, \alpha) \rightarrow (N', \beta)$

then  $\bar{g}_1 - \bar{g}_2 = 0$  so  $\text{im}(g_1 - g_2) \in \mathcal{B}$

Let  $N'' = N' / \text{im}(g_1 - g_2)$  and  $\bar{N}' \xrightarrow{\cong} \bar{N}'' = \bar{N}' / \text{im}(\bar{g}_1 - \bar{g}_2)$

Then

$(N, \alpha) \rightrightarrows (N', \beta) \rightarrow (N'', \gamma)$  is a coequalizer.

We define a functor

$$\mathbb{I}_L \longrightarrow \text{Cat} \quad (\text{category of small categories})$$

$$(N, \alpha) \longmapsto \Sigma_N$$

$$(N, \alpha) \longmapsto (N', \beta) \longmapsto \Sigma_N \longrightarrow \Sigma_{N'}$$

$$g: N \rightarrow N' \\ \beta \circ g \circ \alpha = \alpha$$

$$(N, h: A \rightarrow N) \longmapsto (N', g \circ h)$$

We also define functors

$$P_{(N, \alpha)}: \Sigma_N \longrightarrow F_L$$

for fixed  $L$  and arbitrary  $(N, \alpha) \in \mathbb{I}_N$

by

$$(N, h: M \rightarrow N) \longmapsto (M, \bar{h} \circ \alpha^{-1}: L \xrightarrow{\cong} \bar{N} \xrightarrow{\cong} \bar{M})$$

and for maps  $g: (N, \alpha) \rightarrow (N', \beta)$

$$P_{(N', \beta)} \circ g_* = P_{(N, \alpha)}.$$

So there is a map

$$\text{colim}_{(N, \alpha) \in \mathbb{I}_N} \Sigma_N \longrightarrow F_L.$$



Step 2 The map

$$\text{colim}_{(N, \alpha) \in I_N} \Sigma_N \rightarrow F_L$$

is a isomorphism of categories.

We just check on objects because the argument on morphisms is basically the same.

$$\text{Note that } (M, \theta: L \xrightarrow{\cong} \bar{M}) = P_{(M, \theta^{-1})}^{(M, \text{id}_M)}$$

for any  $(M, \theta)$  in  $F_L$ . So this

map is surjective on objects. Given

$$P_{(N, \alpha)}^{(N, h)} = P_{(N, \alpha)}^{(N', h')}$$

then  $N = N'$  and  $\bar{h} = \bar{h}'$ . Letting

$N' = \text{im}/(h-h')$  we produce map

$$(N, \alpha) \rightarrow (N', \beta) \text{ s.t. } g_\alpha(N, h) = g_\beta(N, h')$$

So as objects in  $\text{colim}_{(N, \alpha) \in I_N} \Sigma_N$   $(N, h) = (N', h')$ .

Step 3. We already constructed  $\Sigma_N^1$  and functors  $k_N^1: \Sigma_N^1 \rightarrow \mathcal{QB}$ .

We again apply Quillen's theorem A.

It suffices to show  $B/(k_N^1/B) \simeq *$

for all objects  $B \in \mathcal{B}$ . Note that

$k_N^1/B$  is fibered over  $\Sigma_N^1$

and it has objects  $((m, h), u)$

where  $(m, h) \in \Sigma_N^1$  and  $u: B \rightarrow \ker h$

is a map in  $\mathcal{QB}$ . Let  $\mathcal{C}$  be

the full subcategory of  $k_N^1/B$  consisting

of  $((m, h), u)$  such that  $u$  is of

the form  $j! : B \twoheadrightarrow \ker h$ .

Since every map  $u$  in  $\mathcal{QB}$  is of

the form  $u = j! \circ i!$  for some epi  $j$  and mono  $i!$ ,

Given  $((M, h), u) \in \text{ker}h/B$

We write

$$u: \text{ker}h \xrightarrow{w} B$$

$$\begin{array}{ccc} \text{ker}h & \xrightarrow{w} & B \\ \parallel & \searrow & \downarrow \\ \text{ker}h & \xrightarrow{j} & B_0 \\ & & \parallel \\ & & B \end{array}$$

So  $u = j \circ i$

and define

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{ker}h & \rightarrow & M & \xrightarrow{h} & N \rightarrow 0 \\ & & \downarrow i & & \downarrow & & \parallel \\ 0 & \rightarrow & B_0 & \xrightarrow{i} & i_* M & \xrightarrow{\tilde{h}} & N \rightarrow 0 \end{array}$$

Then  $((M, h), u) \xrightarrow{i} ((i_* M, \tilde{h}), j)$

is adjoint to the inclusion

$$\mathcal{C} \hookrightarrow \text{ker}h/B$$

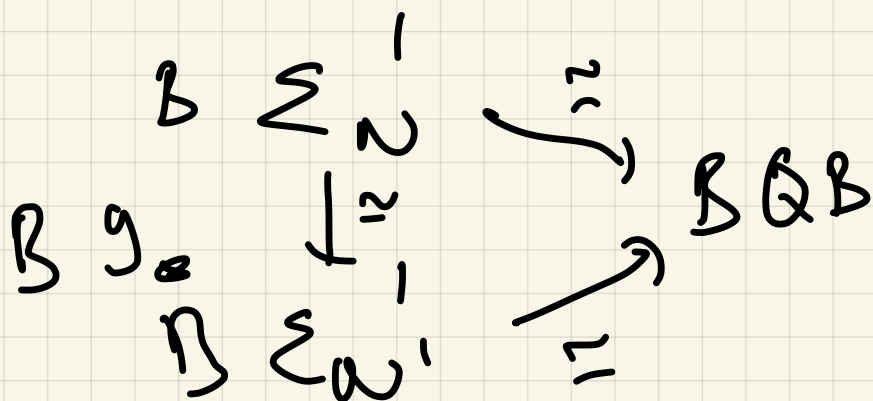
$$j_B: B \rightarrow 0$$

Moreover,  $((N, id_N), j_B)$

is initial in  $\mathcal{C}$  so  $B\mathcal{C} \cong *$ .

This is homotopy equivalence

is compatible w/  $g_0 |_{\Sigma_N'}$



We now prove  $\Sigma_N' \hookrightarrow \Sigma_N$

induces a homotopy equivalence  $\mathcal{B} \Sigma_N' \simeq \mathcal{B} \Sigma_N$

Let  $\mathcal{I}$  be the ordered set of subobjects

$I \hookrightarrow N$  of  $N$  s.t.  $N/I \in \mathcal{B}$ .

regarded as a category. We observe

that there is a functor

$$P_N: \Sigma_N \rightarrow \mathcal{I}$$

$$(m, n) \longmapsto m \hookrightarrow n$$

which is fibered with fiber  $P_N^{-1}(I) = \Sigma_I'$ .

The base change functor

$$P_N^{-1}(I \rightarrow J): P_N^{-1}(I) \rightarrow P_N^{-1}(J)$$

$$\cong \sum_I \rightarrow \sum_J$$

$$(M, n \rightarrow I) \longmapsto (J \times_M I, J \times_M I \rightarrow J)$$

This clearly commutes with

the inclusions

$$\begin{array}{ccc} \sum_I & \rightarrow & \sum_J \\ & \searrow & \swarrow \\ & \sum_N & \end{array}$$

By the previous result

$$\begin{array}{ccc} B \sum_I & \rightarrow & B \sum_J \\ \downarrow & & \downarrow \\ BQI & & BQJ \end{array}$$

So we can apply Quillen's theorem  $B$ .

Also,  $B I \simeq 0$  since  $I$  is a filtered partially ordered set. So  $B \sum_N \xrightarrow{\sim} B \sum_N$ .

Note that these are also compatible with the maps

$$\begin{array}{ccc}
 P_{(N, \alpha)}: \Sigma_N & \longrightarrow & F_L \\
 \downarrow & & \downarrow \\
 P_{(M, \beta)}: \Sigma_M & \longrightarrow & F_{L'}
 \end{array}$$

So

$$\begin{array}{ccccc}
 QB \cong & B_{\text{col.}} \Sigma_N & \xrightarrow{\cong} & B F_L & \xrightarrow{\cong} & B_{L'} Q_{\text{loc}} \\
 & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 QB \cong & B_{\text{col.}} \Sigma_M & \xrightarrow{\cong} & B F_{L'} & \xrightarrow{\cong} & B_{L'} Q_{\text{loc}}
 \end{array}$$

□

By Quillen,

$p$  prime

$$K_i(\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/(p^k-1) & i=2k-1 > 0 \\ 0 & i=2k \end{cases}$$

So if you know  $K_i(\mathbb{Q})$ , then

you can compute  $K_i(\mathbb{Z})$  by

the LES

$$\hookrightarrow \prod_p K_i(\mathbb{Z}/p) \rightarrow K_i(\mathbb{Z}) \rightarrow K_i(\mathbb{Q})$$

$$\hookrightarrow \prod_p K_{i-1}(\mathbb{Z}/p) \rightarrow K_{i-1}(\mathbb{Z}) \rightarrow K_{i-1}(\mathbb{Q})$$

There is an injection

$$\hookrightarrow \dots \quad \text{Ex: } K_{2i}(\mathbb{Z}) \twoheadrightarrow K_{2i}(\mathbb{Q})$$

$i \geq 0$