# Lecture 5:

# The +-construction

# I. Motivation

Kervaire originally defined the construction $M^+$ as a way to construct a homotopy sphere from a homology sphere $M$. Suppose $M$ is a smooth manifold and

$$H_*(M;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z}) \quad n \geq 3.$$

Then we know $\pi_1 M$ is a perfect group $\pi_1 M$ and $M^+$ has the property that $\pi_1 M^+ = 0$ but $H_*(M;\mathbb{Z}) \cong H_*(M^+;\mathbb{Z})$ so $M^+$ is a homotopy $n$-sphere.

## II. Quillen's + -construction K-theory

We defined a group $GL(R)$ and last time we defined its classifying space $BGL(R)$ which has the property that

1) $BGL(R)$ is a CW complex
2) $\pi_k BGL(R) = \begin{cases} GL(R) & k=1 \\ 0 & \text{otherwise} \end{cases}$

In fact, $BGL(R)$ is the unique space up to homotopy equivalence with these properties.

The construction $BGL(R)^+$ will have the property that

$$\pi_1 BGL(R)^+ \cong \pi_1 BGL(R)/N$$

where $N$ is a perfect normal subgroup. In our case, $N = E(R)$

so $\pi_1 BGL(R)^+ = GL(R)/E(R) = K_1(R)$

by construction.

<u>Def</u>: Let $R$ be a ring. We define the algebraic K-theory space of $R$ by

$$K(R) := K_0(R) \times BGL(R)^+ .$$

Moreover, $K_n(R) = \pi_n K(R)$.

The $+$-construction of a based path connected space $X$ has the following properties:

1) $\pi_1 X^+ \cong \pi_1 X / N$

for some perfect normal subgroup $N$ of $\pi_1 X$

2) The map

$$H_*(X, L) \longrightarrow H_*(X^+, L)$$

is an isomorphism for any $\mathbb{Z}[\pi_1 X]$-module $L$.

Here $H_*(X, L) := H_*(L \otimes_{\mathbb{Z}[\pi_1 X]} C_*(\tilde{X}; \mathbb{Z}))$

where $\tilde{X}$ is the universal cover of $X$.

Construction: We will form a relative CW complex $(X^+, X)$ by attaching only 2-cells and 3-cells to $X$.

First, let $\alpha \in I$ be a minimal set of generators for $N < \pi_1 X$ where $N$ is a perfect normal subgroup of $\pi_1 X$. Let $f_\alpha : S^1 \longrightarrow X$ be a representative for $\alpha \in N \subseteq \pi_1 X$. Then form the pushout

$$\begin{tikzcd} \coprod_{\alpha \in I} S^1 \arrow[r, "\amalg f_\alpha"] \arrow[d] & X \arrow[d] \\ \coprod_{\alpha \in I} D^2 \arrow[r] & X_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

to define $X_1$.

Then

$$\pi_1 X_1 \cong \pi_1 X / N \quad \text{by construction}$$

In homology, we have

$$H_2(X_1;\mathbb{Z}) \to H_1(\coprod_{\alpha\in I} S^1;\mathbb{Z}) \to H_1(X;\mathbb{Z}) \xrightarrow{\cong} H_1(X_1;\mathbb{Z})$$

So $X_1$ does not have the desired property. We then pass to the universal cover $\tilde{X}_1$ and take a pullback

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X}_1 \arrow[d] \\ X \arrow[r] & X_1 \end{tikzcd}$$

then $\hat{X}$ will be a Galois covering corresponding to $N \lhd \pi_1 X$ with associated Galois group $\pi_1 X / N$.

$$\begin{tikzcd} \hat{X} \arrow[r, hook] \arrow[d] & \tilde{X}_1 \arrow[d, "\pi"] \\ X \arrow[r, hook] & X_1 \end{tikzcd}$$

for each 2-cell $a_\alpha$ of the relative CW complex $(X_1, X)$ there is a collection of 2-cells $\pi^{-1}(a_\alpha)$ of $(\tilde{X}_1, \hat{X})$. We know $\pi_1 X$ acts transitively on the 2-cells with stabilizer $\pi_1 X/N$, so

$$C_2(\tilde{X}_1, \hat{X}; \mathbb{Z}) \cong H_2(\tilde{X}_1, \hat{X}; \mathbb{Z})$$

is a free $\mathbb{Z}[\pi_1 X/N]$-module with generators $[\tilde{a}_\alpha]$ with $\tilde{a}_\alpha$ a lift of $a_\alpha$.

We then form a diagram

$$\begin{tikzcd}
\pi_2 \hat{X} \arrow[r] \arrow[d] & \pi_2 \tilde{X}_1 \arrow[r] \arrow[d] & \pi_2 (\tilde{X}_1, \hat{X}) \arrow[r] \arrow[d] & \pi_1 \hat{X} \arrow[d] \\
H_2(\hat{X};\mathbb{Z}) \arrow[r] & H_2(\tilde{X}_1;\mathbb{Z}) \arrow[r, two heads] & H_2(\tilde{X}_1, \hat{X};\mathbb{Z}) \arrow[r] & H_1(\hat{X};\mathbb{Z}) \arrow[d, phantom, "\shortparallel"] \\
 & & & 0
\end{tikzcd}$$

where the vertical maps are the Hurewicz homomorphism and the rows are the long exact sequence for a pair $(\tilde{X}_1, \hat{X})$. We know

$H_1(\hat{X};\mathbb{Z}) \cong N/[N,N] = 0$ since $N$ is perfect. Also, $\pi_1 \tilde{X}_1 = 0$ so $\pi_2(\tilde{X}_1) \cong H_2(\tilde{X}_1;\mathbb{Z})$. Thus, there is a surjection

$$\pi_2 \tilde{X}_1 \twoheadrightarrow H_2(\tilde{X}_1;\mathbb{Z}) \twoheadrightarrow H_2(\tilde{X}_1, \hat{X};\mathbb{Z}).$$

For each $[\tilde{\alpha}_\alpha] \in H_2(\tilde{X}_1, \hat{X}; \mathbb{Z})$,

choose a lift $[\tilde{f}_\alpha] \in \pi_2 \tilde{X}_1$

represented by

$$\tilde{f}_\alpha : S^2 \longrightarrow \tilde{X}_1$$

and let

$$g_\alpha : S^2 \longrightarrow \tilde{X}_1 \xrightarrow{\pi} X_1 .$$

Then we attach 3-cells via

$$\begin{tikzcd}
\coprod_{\alpha \in I} S^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_1 \arrow[d] \\
\coprod_{\alpha \in I} D^3 \arrow[r] & X^+
\end{tikzcd}$$

to form $X^+$.

Now we have to check the desired properties.

Since we only attached 3-cells to $X_1$ to form $X^+$, we still have

$$\pi_1 X^+ \cong \pi_1 X_1 = \pi_1 X / N.$$

For the second property, consider the square of pullbacks

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X}^+ \arrow[d] \\ X \arrow[r] & X_1 \arrow[r] & X^+ \end{tikzcd}.$$

Then $C_*(\tilde{X}^+, \hat{X}; \mathbb{Z})$ is concentrated in degrees 2 and 3

$$0 \to C_3(\tilde{X}^+, \hat{X}; \mathbb{Z}) \xrightarrow{d} C_2(\tilde{X}^+, \hat{X}; \mathbb{Z}) \to 0$$

where $C_3(\tilde{X}^+, \hat{X}; \mathbb{Z}) \cong H_3(\tilde{X}^+, \tilde{X}_1; \mathbb{Z})$

$$C_2(\tilde{X}^+, \hat{X}; \mathbb{Z}) \cong H_2(\tilde{X}_1, \hat{X}; \mathbb{Z}).$$

and the boundary map $\partial$ is exactly the boundary map

$$H_3(\tilde{X}^+, \tilde{X}_1) \longrightarrow H_2(\tilde{X}_1, \hat{X}; \mathbb{Z})$$

for the long exact sequence of the triple $(\tilde{X}^+, \tilde{X}_1, \hat{X})$. By construction, this map factors as

$$\begin{tikzcd} H_3(\tilde{X}^+, \tilde{X}_1; \mathbb{Z}) \arrow[dr, "\partial"'] & & H_2(\tilde{X}_1, \hat{X}; \mathbb{Z}) \\ & H_2(\tilde{X}_1; \mathbb{Z}) \arrow[ur, "j"'] & \end{tikzcd}$$

where $\partial$ is the boundary map for the long exact sequence of the pair $(\tilde{X}_1, \tilde{X}^+)$ and $j$ is induced by the canonical map of pairs $(\tilde{X}_1, \emptyset) \to (\tilde{X}_1, \hat{X})$.

We consider the diagram

$$\begin{tikzcd}
\pi_3(X^+) \arrow[r] \arrow[d] & \pi_3(\tilde{X}^+, \tilde{X}_1) \arrow[r] \arrow[d] & \pi_2 \tilde{X}_1 \arrow[d] \\
H_3(\tilde{X}^+; \mathbb{Z}) \arrow[r] & H_3(\tilde{X}^+, \tilde{X}_1; \mathbb{Z}) \arrow[r, "d"] & H_2(\tilde{X}_1; \mathbb{Z})
\end{tikzcd}$$

where again vertical maps are the Hurewicz homomorphism and rows are long exact sequences of the pair $(\tilde{X}^+, \tilde{X}_1)$.

Let $[\tilde{b}_\alpha]$ be a 3-cell of $(\tilde{X}^+, \tilde{X}_1)$ in $C_3(\tilde{X}^+, \tilde{X}_1)$

$$\cong H_3(\tilde{X}^+, \tilde{X}_1)$$

representing a basis element for the free $\mathbb{Z}[\pi_1 X]$-module $H_3(\tilde{X}^+, \tilde{X}_1)$.

Similarly, let $\tilde{f}_\alpha : S^2 \to \tilde{X}_1$ be

representive for a class $[\tilde{f}_\alpha] \in \pi_2 \tilde{X}_1$

corresponding to the attaching map

$S^2 \to \tilde{X}_1$ for forming $\tilde{X}^+$ from $\tilde{X}_1$.

lifting $f_\alpha : S^2 \to \tilde{X}_1$. Then

by construction

$$H_3(\tilde{X}^+, \tilde{X}_1; \mathbb{Z}) \xrightarrow{\partial} H_2(\tilde{X}_1; \mathbb{Z})$$

sends $\partial([b_\alpha]) = [f_\alpha]$.

Moreover, the map

$$H_2(\tilde{X}_1; \mathbb{Z}) \xrightarrow{j} H_2(\tilde{X}_1, \hat{X}; \mathbb{Z})$$

sends $[f_\alpha]$ to $[\tilde{\alpha}_\alpha]$, which is a basis elt. for $H_2(\tilde{X}_1, \hat{X}; \mathbb{Z})$ as a free $\mathbb{Z}[\pi_1 X]$-module.

Since these are both free $\mathbb{Z}[\pi,X]$-modules w/ basis indexed by the same set $I$, we have shown the map

$$d: C_3(\tilde{X}^+, \hat{X}; \mathbb{Z}) \to C_2(\tilde{X}, \hat{X}; \mathbb{Z})$$

is an isomorphism so

$C_\bullet(\tilde{X}^+, \hat{X}; \mathbb{Z})$ is acyclic

and $L \otimes_{\mathbb{Z}[\pi,X]} C_\bullet(\tilde{X}^+, \hat{X}; \mathbb{Z})$ remains acyclic. Thus,

$$H_\bullet(X^+, X; L) \cong 0 \quad \text{as desired.}$$

Example:

Let
$$\begin{array}{ccc} \Sigma_n & \to & \Sigma_{n+1} \\ \| & & \| \\ \mathrm{Aut}(\{1,\dots,n\}) & & \mathrm{Aut}(\{1,\dots,n+1\}) \end{array}$$

$$\sigma \longmapsto \sigma'(s) = \begin{cases} \sigma(s) & s \le n \\ n+1 & s = n+1 \end{cases}$$

The define
$$\Sigma = \bigcup_{n \ge 1} \Sigma_n$$

Similarly, let $A_n \subseteq \Sigma_n$ be the alternating group and define

$$A = \bigcup_{n \ge 1} A_n.$$ Then

$A < \Sigma$ is a perfect normal subgroup and $\Sigma/A = \mathbb{Z}/2$.

So $\pi_1(B\Sigma^+) \cong \mathbb{Z}/2$.

Thm [Barratt-Priddy-Quillen]

There is an isomorphism

$$\pi_k^S := \operatorname{colim}_k \pi_{n+k} S^n \cong \pi_k(\mathbb{Z} \times B\Sigma^+).$$

Example:

Consider $BGL(R)$ with perfect normal subgroup $E(R) < \pi_1 GL(R)$.

Then

$$K(R) = K_0(R) \times BGL(R)^+.$$

## II The $+$-construction as group completion.

In fact, $BGL(R)^+$ is a commutative $H$-group and it has the universal property that given an $H$-space $Y$ with a map $BGL(R) \to Y$, then there exists an extension

$$\begin{tikzcd} BGL(R) \arrow[r] \arrow[dr] & BGL(R) \arrow[d, dashed, "h"] \\ & Y \end{tikzcd}$$

Def: An H-space is a topological space $X$ with a continuous map

$$\mu : X \times X \to X$$

and a unit map

$$e : * \to X$$

such that the diagram

$$\begin{tikzcd}
X \times * \arrow[r, "X \times e"] \arrow[dr, "id"'] & X \times X \arrow[d, "\mu"] & * \times X \arrow[l, "e \times X"'] \arrow[dl, "id"] \\
& X &
\end{tikzcd}$$

commutes in $ho(Top)$, in other words

$$\mu(-,e) \simeq id \simeq \mu(e,-).$$

Example:

$\Omega Y$ is an H-space with

$$* : \Omega Y \times \Omega Y \longrightarrow \Omega Y$$

given by concatenation of loops.

When $X$ is an H-space, $\pi_1 X$ is an abelian group by the Eckmann-Hilton argument.

<u>Def</u>: A commutative H-group is an H-space w/ an inverse map $i : X \to X$ such that $(X, m, e, i)$ forms a commutative group object in $ho(Top)$.

# Obstruction theory

Let $(X,A)$ be a relative CW complex with finitely many cells s.t. $A$ and $X$ are based and path connected. Consider the following

questions: Given a map $f: A \to Y$ where $Y$ is path connected

1) When can we extend $f$ to $F$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \\ X \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

commutes?

2) When is this choice unique up to homotopy $H: X \times I \to Y$ rel $A$?

The idea is to extend inductively up the skeleta

$$\begin{tikzcd}
\coprod_{\alpha \in I_1} S^0 \arrow[r] & A \arrow[r, "f"] \arrow[d, hook] & Y \\
\coprod_{\alpha \in I_2} S^1 \arrow[r] & X_1 \arrow[d, hook] \arrow[ur, dashed, "F_1"] & \\
\coprod_{\alpha \in I_2} S^2 \arrow[r] & X_2 \arrow[d, hook] \arrow[uur, dashed, "F_2"] & \\
& \vdots \arrow[d, hook] & \\
& X_n \arrow[uuuur, dashed, "F_n"'] &
\end{tikzcd}$$

where $X_k$ is the $k$-skeleton of our relative CW complex $A$.

Since $Y$ is path connected $\pi_0 Y \cong 0$ and $F_1$ can be any null homotopic map s.t. $F_1|_A = f$.

To construct $F_2$, it turns out the obstruction to extending is detected by $\pi_1$ in that we can extend to $X_2$ iff there exists a group homomorphism $\theta : \pi_1 X_2 \to \pi_1 Y$ such that

$$(*) \quad \begin{tikzcd} \pi_1 X_1 \arrow[r] \arrow[d] & \pi_1 Y \\ \pi_1 X_2 \arrow[ur, "\theta"'] & \end{tikzcd}$$

commutes.

Next, suppose we have exended to $X_2 \xrightarrow[F_2]{} Y$ so there exists

$$\theta : \pi_1 X_2 \to \pi_1 Y$$

such that $(*)$ commutes.

Then the obstruction to lifting

further to $X_n$ lie in

$$H^{n+1}(X_n, A, \theta^* \pi_n Y)$$

where we use the fact that

$\pi_n Y$ is a $\mathbb{Z}[\pi_1 Y]$-module

for $n \geq 1$ and consequently a

$\mathbb{Z}[\pi_1 X]$-module via restriction.

along

$$\mathbb{Z}[\pi_1 X] \xrightarrow{\mathbb{Z}[\theta]} \mathbb{Z}[\pi_1 Y]$$

denoted $\theta^* \pi_n Y$.

Example: We want extensions

$$\begin{tikzcd}
BGL(R) \arrow[rr, "f"] \arrow[d] & & Y \\
BGL(R)^{+}_{1} \arrow[urr, dashed, "F_1"] \arrow[d] & & \\
BGL(R)^{+} \arrow[uurr, dashed, "F_2"'] & &
\end{tikzcd}$$

The extension $F_1$ exists because of the universal property of $(\pi_1 BGL(R))^{ab}$

abelian

$$\begin{tikzcd}
GL(R) \arrow[r] \arrow[d] & \pi_1 Y \\
GL(R)^{ab} \arrow[ur, dashed, "\Theta"'] &
\end{tikzcd}$$

The obstructions to extending

$$\begin{tikzcd} BGL(R)_1^+ \arrow[r] \arrow[d] & Y \\ BGL(R)^+ \arrow[ur, dashed] & \end{tikzcd}$$

lie in $H^{n+1}(BGL(R)_1^+, BGL(R); \theta^* \pi_n Y)$

but these groups vanish as we saw earlier.

We also want to know about uniqueness. We return to the general setup.

Given two extensions

$F, F' : X_n \to Y$ s.t.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \\ X_n \arrow[ur, "F, F'"'] & \end{tikzcd}$$

commute.

Then we define a map

$$X \times \{0,1\} \cup A \times I \xrightarrow{\tilde{f}} Y$$

by $\tilde{f}|_{X \times \{0\}} = F$ $\quad$ $\tilde{f}|_{A \times I} = f \circ \pi_1$

$\tilde{f}|_{X \times \{1\}} = F'$ $\quad$ where

$\pi_1 : A \times I \to A$

is projection.

The question of uniqueness is then the following question:

Can we extend

$$\begin{tikzcd}
X \times \{0,1\} \cup A \times I \arrow[r, "\tilde{f}"] \arrow[d] & Y \\
X \times I \arrow[ur, "H"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \pi_1 X \arrow[d] \arrow[dr, bend left, "\theta_2"] & \\
\pi_1 A \arrow[ur] \arrow[dr] & \langle \pi_1 X *_{\pi_1 A} \pi_1 X \rangle \arrow[r] & \pi_1 Y \\
 & \pi_1 X \arrow[u] \arrow[ur, "\theta_1"'] &
\end{tikzcd}$$

The first obstruction $\pi_1 X \xrightarrow{\theta_1} \pi_1 Y$
$\pi_1 X \xrightarrow{\theta_2} \pi_1 Y$

$$\begin{tikzcd}
\pi_1(X \times \{0,1\} \cup A \times I) \arrow[r] \arrow[d, "\text{quotient}"'] & \pi_1 Y \\
\pi_1 X \arrow[ur, dashed] &
\end{tikzcd}$$

$\cong$ $\pi_1 X *_{\pi_1 A} \pi_1 X$

$\cap$

$(\theta_1, \theta_2) \circ \Delta = \theta'$

always vanishes.

The remaining obstructions lie in

$$H^{n+2}\left(X \times I,\ X \times \{0,1\} \cup A \times I\ ;\ \theta'^{*} \pi_{n+1} Y\right)$$

$$\cong$$

$$H^{n+1}\left(X,\ A\ ;\ \theta^{*} \pi_{n+1} Y\right)$$

suspension isomorphism.

In our case,

$$H^{n+1}\left(BGL(R)^{+},\ BGL(R)\ ;\ \theta^{*} \pi_{n+1} Y\right)$$

$$\cong 0 \quad \forall\ n \geq 1$$

So the extension

$$\begin{tikzcd} BGL(R) \arrow[r] \arrow[dr, bend right] & BGL(R)^{+} \arrow[d, dashed] \\ & Y \end{tikzcd}$$

is unique up to htpy rel $BGL(R)$.

We won't give a complete proof that $BGL(\mathbb{R})^+$ is a commutative H-group, but let's define the operation. Let

$$\mu_0 : GL(\mathbb{R}) \times GL(\mathbb{R}) \to GL(\mathbb{R})$$

be defined for $A = (a_{ij}), B = (b_{ij}) \in GL(\mathbb{R})$ by

$$\mu_0(A, B) = (c_{ij})$$

where

$$c_{ij} = \begin{cases} a_{st} & \text{if } i = 2s-1 \\ & \quad\; j = 2t-1 \\ b_{st} & \text{if } i = 2s \quad j = 2s \\ 0 & \text{o.w.} \end{cases}$$

Ex:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A \qquad \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} = \mu_0(A, B)$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = B$$

Then $\mu_0$ induces a map

$$\begin{tikzcd} B(GL(R) \times GL(R)) \arrow[r, "B\mu_0"] & BGL(R) \\ BGL(R) \times BGL(R) \arrow[u, "\simeq"'] & \end{tikzcd}$$

One can then show that if $N_1 < \pi_1 X$, $N_2 < \pi_1 Y$ are perfect normal subgroups then $N_1 \times N_2 < \pi_1(X \times Y) = \pi_1 X \times \pi_1 Y$ is a perfect normal subgroup and there is a homotopy equivalence

$$X^+ \times Y^+ \xrightarrow{\simeq} (X \times Y)^+$$

So we define $\mu$ by

$$\begin{tikzcd} BGL(R)^+ \times BGL(R)^+ \arrow[r, "\mu"] \arrow[d, "\wr"] & BGL(R)^+ \\ (BGL(R) \times BGL(R))^+ \arrow[r, "\cong"] & B(GL(R) \times GL(R))^+ \arrow[u, "(B\mu_0)^+"'] \end{tikzcd}$$

One can check that this gives

$$BGL(R)^+$$

the structure of a commutative H-group. Also,

$K_0(G)$ is a discrete abelian group so

$K_0(G) \times BGL(R)^+$ is a commutative H-group.

By the recognition theorem, $K_0(G) \times BGL(R)^+$ is an infinite loop space so it can be regarded as an $\Omega$-spectrum.