Lecture 11: The fibration Theorem

In order to prove the fibration theorem, we will need some extra hypotheses on our Waldhausen category.

<u>Def.</u> We say a Waldhausen category $\mathcal{C}$ has a cylinder functor if it is equipped with a functor

$$T : \operatorname{Arr}\mathcal{C} \longrightarrow \mathcal{C}$$

and natural transformations satisfying

$$\begin{tikzcd}
s(-) \arrow[r, "j_1"] \arrow[dr] & T(-) \arrow[d, "p"] & t(-) \arrow[l] \arrow[dl, equal] \\
& t(-) &
\end{tikzcd}$$

where $s : \operatorname{Arr}(\mathcal{C}) \to \mathcal{C}$, $(A \to B) \mapsto A$, $t : \operatorname{Arr}\mathcal{C} \to \mathcal{C}$, $(A \to B) \mapsto B$ and $\underset{A}{\overset{\shortparallel}{s(f)}} \xrightarrow{f} \underset{B}{\overset{\shortparallel}{t(f)}}$.

Additionally, we ask that our cylinder functor satisfies

1) There is an exact functor

$$\operatorname{Arr}(\mathcal{C}) \longrightarrow c_1\mathcal{C} \subseteq \operatorname{Arr}\mathcal{C}$$
$$(f : A \to B) \longmapsto A \vee B \xrightarrow[j_1(f) \vee j_2(f)]{} T(A)$$

($c_1\mathcal{C} \subseteq \operatorname{Arr}\mathcal{C}$ full subcategory on cofibrations)

2) We have $T(0 \to A) = A$ for each A in $\mathcal{C}$ and

$$j_1(0 \to A) = p(0 \to A) = id_A.$$

Additionally, we say the cylinder functor T satisfies the cylinder axiom if

$$p(f) : T(f) \to t(f) \in w_1\mathcal{C} \subseteq \operatorname{Arr}\mathcal{C}$$

for all f in $\operatorname{Arr}\mathcal{C}$

($w_1\mathcal{C}$ full subcategory on weak equiv.)

Example:

The Waldhausen category $R^f(X)$ with $wR^f(X)$ the homotopy equivalences has a cylinder functor

$$T(f: Y \to Y') = X \times_{X \times [0,1]} Y \times [0,1] \times_{Y \times \{1\}} Y'$$

satisfying the cylinder axiom.

When $R^f(X)$ is equipped with $wR^f(X) = iso R^f(X)$, i.e the same category with cofibrations but weak equivalences are homeomorphisms, then the cylinder functor still exists, but it doesn't satisfy the cylinder axiom.

Ex: Let $\mathcal{E}$ be an exact category and consider the Waldhausen category $Ch(\mathcal{E})$ of chain complexes in $\mathcal{E}$ where $c Ch \mathcal{E}$ consists of levelwise admissible monomorphisms, we fix an embedding $\mathcal{E} \subseteq A$ where A is an abelian category and $w Ch \mathcal{E}$ are maps which are quasi-isomorphisms $Ch A$. We define $Ch^b(\mathcal{E})$ to be the full exact sub Waldhausen category on the bounded chain complexes in $\mathcal{E}$.

Then $Ch^b(\mathcal{E})$ has a cylinder functor

$$T(f: C_\cdot \to C_\cdot')_n = C_n \oplus C_{n-1} \oplus C_n'.$$

Def: Given a Waldhausen category $\mathcal{C}$ and a cylinder functor $T: \mathrm{Arr}\,\mathcal{C} \to \mathcal{C}$ we can define the cone as the composite

$$C: \mathcal{C} \longrightarrow \mathrm{Arr}\,\mathcal{C} \xrightarrow{T} \mathcal{C}$$
$$A \longmapsto (A \to 0) \longmapsto T(c,0)$$

so $C(A) = T(A \to 0)$.

we then define
$$\Sigma: \mathcal{C} \longrightarrow \mathcal{C}$$
to be the cofiber of the natural transformation

$$\mathrm{cof}\,(\mathrm{id}(-) \rightarrowtail C(-)) = \Sigma(-).$$

Ex: In $Ch(\mathcal{C})$, $\Sigma(C_\bullet) = C_\bullet[-1]$

with $(\Sigma C_\bullet)_n = C_{n-1}$

Def: We say a Waldhausen category $(\mathcal{C}, c\mathcal{C}, w\mathcal{C})$ satisfies the saturation axiom if $w\mathcal{C}$ satisfies 2 out of three; i.e for all composable pairs $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ such that 2 out of three of $\{f, g, g\circ f\}$ are in $w\mathcal{C}$ then so is the third.

Lemma 1 If $\mathscr{C}$ is a Waldhausen category with a cylinder functor T then $S_n\mathscr{C}$ has a cylinder functor

$$T' = S_n T : \mathrm{Arr}(S_n\mathscr{C}) = S_n \mathrm{Arr}\mathscr{C} \to S_n\mathscr{C}$$

w/ natural transformations $j_1' = S_n j_1$, $j_2' = S_n j_2$, and $p' = S_n p$ satisfying

$$\begin{tikzcd} s(-) \arrow[r, "j_1'"] \arrow[dr] & T' \arrow[d, "p'"] & t(-) \arrow[l, "j_2'"'] \arrow[dl, equal] \\ & t(-) & \end{tikzcd}$$

If T satisfies the cylinder axiom, then so does T'.

If $\mathscr{C}$ satisfies the saturation axiom, then so does $S_n\mathscr{C}$.

__Proof.__ Exercise

__Def.__ We say a Waldhausen category $\mathscr{C}$ satisfies the extension axiom if for each map of cofiber sequences

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

such that $A \to A'$ and $C \to C'$ are weak equivalences then $B \to B'$ is also a weak equivalence.

Thm. (Fibration theorem)

Let $(\mathscr{C}, c\mathscr{C})$ be a category with cofibrations equipped with two subcategories $v\mathscr{C} \subseteq w\mathscr{C}$ of weak equivalences such that $(\mathscr{C}, c\mathscr{C}, v\mathscr{C})$ and $(\mathscr{C}, c\mathscr{C}, w\mathscr{C})$ are Waldhausen categories. In addition, assume $(\mathscr{C}, c\mathscr{C}, w\mathscr{C})$ has a cylinder functor satisfying the cylinder axiom and $w\mathscr{C}$ satisfies the saturation axiom and the extension axiom. Let $(\mathscr{C}^w, c\mathscr{C}^w, v\mathscr{C}^w)$ denote the full sub Waldhausen category of $(\mathscr{C}, c\mathscr{C}, v\mathscr{C})$ on objects such that $0 \rightarrowtail A$ is a map in $w\mathscr{C}$. Then there is a fiber sequence

$$\begin{array}{ccccc} K(\mathscr{C}^w, c\mathscr{C}^w, v\mathscr{C}^w) & \to & K(\mathscr{C}, c\mathscr{C}, v\mathscr{C}) & \to & K(\mathscr{C}, c\mathscr{C}, w\mathscr{C}) \\ \| & & \| & & \| \\ K(\mathscr{C}^w) & & K(\mathscr{C}, v) & & K(\mathscr{C}, w) \end{array}$$

and consequently a long exact sequence

$$\cdots \to K_i(\mathscr{C}^w) \to K_i(\mathscr{C}, v) \to K_i(\mathscr{C}, w) \to$$

$$\to K_{i-1}(\mathscr{C}^w) \to K_{i-1}(\mathscr{C}, v) \to K_{i-1}(\mathscr{C}, w) \to \cdots$$

$$K_0(\mathscr{C}^w) \to K_0(\mathscr{C}, v) \to K_0(\mathscr{C}, w) \to 0.$$

To prove the theorem, we need a preliminary discusion on bicategories.

Note: We can identity small categories with their essential image in sSet via the functor $N_\bullet : \mathrm{Cat} \to \mathrm{sSet}$. We will build this into the definition of bicategories

<u>Def</u>: A **bicategory** is a bisimplicial set

$$\mathcal{B}_{\bullet\bullet} : \Delta^{op} \times \Delta^{op} \longrightarrow \mathrm{Set}$$

such that $\mathcal{B}_{p\bullet}$ and $\mathcal{B}_{\bullet q}$ are each the nerve of a category for each $[p],[q] \in \Delta^{op}$. We call

$\mathcal{B}_{0,0}$ = objects of $\mathcal{B}$ $\qquad$ $\mathcal{B}_{01}$ = vertical morphisms

$\mathcal{B}_{1,0}$ = horizontal morphisms $\mathcal{B}_{11}$ = bimorphisms

Ex: Given a category B, we can form $\mathrm{bi}B$ with

$$\mathrm{bi}B = \mathrm{ob}B$$
$$(\mathrm{bi}B)_{10} = (\mathrm{bi}B)_{01} = \mathrm{Arr}(B)$$
$$(\mathrm{bi}B)_{11} \ni \begin{array}{ccc} a & \to & b \\ \downarrow & & \downarrow \\ a' & \to & b' \end{array}$$ commuting diagram in B

If $A \subseteq B$ is a subcategory write AB for the subbicategory of $\mathrm{bi}B$ with

$AB_0 = \mathrm{ob}\,B = \mathrm{ob}\,A \qquad AB_{01} = \mathrm{Arr}(A) \qquad a \to b, a' \to b' \in \mathrm{Arr}B$

$AB_{10} = \mathrm{Arr}B \qquad AB_{11} \ni \begin{array}{ccc} a & \to & b \\ \downarrow & & \downarrow \\ a' & \to & b' \end{array} \qquad \begin{array}{c} a \\ \downarrow \\ a' \end{array}, \begin{array}{c} b \\ \downarrow \\ b' \end{array} \in \mathrm{Arr}A$

If B is a category, write B for the bicategory with $B_{p,q} = N_p B \;\; \forall q \geq 0$.

Lemma 2 (swallowing lemma) Let $A \subseteq B$ be a subcategory. The map of bicategories

$$B \to AB$$

induces a homotopy equivalence

$$|B| \to |AB|$$

Pf. It suffices to prove that the map

$$N_p B \to AB_{p,\bullet}$$

of simplicial sets induces an equivalence

$$|N_p B| \to |AB_{p,\bullet}|.$$

Define a map

$$AB_{p,\bullet} \to N_p B$$

by $(A_0 \to \dots \to A_n) \mapsto A_0$.

Then clearly

$$\begin{tikzcd} N_p B \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}_{N_p B}"'] & AB_{p,\bullet} \arrow[d, phantom, "="] \arrow[r, "r"] & N_p B \\ & {} & \end{tikzcd}$$

So it suffices to produce a natural transformation

$$\begin{tikzcd} AB_{p,\bullet} \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}_{A_{p,\bullet}}"'] & N_p B \arrow[d, phantom, "\Downarrow \varepsilon"] \arrow[r, "r"] & AB_{p,\bullet} \\ & {} & \end{tikzcd}$$

$\varepsilon : r \circ s \Rightarrow \mathrm{id}_{A_{p,\bullet}}$

We define

$$\eta : A_{p\bullet} \times [1] \longrightarrow AB_{p,\bullet}$$

by
$$\eta(A_0 \to A_1 \to \dots \to A_n, 0) = (A_0 \xrightarrow{id} A_0 \to \dots \to A_0)$$
$$\eta(A_0 \to \dots \to A_n, 1) = (A_0 \to A_1 \to \dots \to A_n)$$

$$\eta(A_0 \to \dots \to A_n, 0 \to 1)$$

$$\|$$

$$\begin{tikzcd}
A_0 \arrow[r, "id"] \arrow[d, "id"'] & A_0 \arrow[r, "id"] \arrow[d, "a_1"] & A_0 \arrow[r] \arrow[d, "a_2 a_1"] & \dots \arrow[r] & A_0 \arrow[d, "a_n \circ \dots \circ a_1"] \\
A_0 \arrow[r, "a_1"] & A_1 \arrow[r, "a_2"] & A_2 \arrow[r, "a_3"] & \dots \arrow[r, "a_n"] & A_n
\end{tikzcd}$$

Thus, for each p there is a homotopy equivalence

$$|N_p B| \xrightarrow[\simeq]{} |AB_{p,\bullet}| .$$

Since

$$B \to AB$$

is a map of bisimplicial sets w/ $|B_{p,\bullet}| \xrightarrow{\simeq} |AB_{p,\bullet}|$ for all $p \geq 0$,

$$\begin{tikzcd}
{|[p] \mapsto |B_{p,\bullet}||} \arrow[r, "\simeq"] \arrow[d, phantom, "\cong" rotate=90] & {|[p] \mapsto |AB_{p,\bullet}||} \arrow[d, phantom, "\cong" rotate=90] \\
{|B|} & {|AB|}
\end{tikzcd} .$$

Proof of fibration theorem.

Recall that we have

$$(\mathcal{C}^w_b, c\mathcal{C}^w_b, v\mathcal{C}^w_b) \subseteq (\mathcal{C}, c\mathcal{C}, v\mathcal{C}) \overset{+}{\subseteq} (\mathcal{C}, c\mathcal{C}, w\mathcal{C})$$

has cylinder functor w/ cylinder axiom, saturation, + extension axioms.

$$v\mathcal{C} \subseteq w\mathcal{C}$$

$$\mathcal{C}^w_b \subseteq \mathcal{C}$$
$$\ni A$$
s.t. $(0 \rightarrowtail A) \in w\mathcal{C}$.

The idea of the proof is to consider the square

$$\begin{tikzcd}
vS_\bullet\mathcal{C}^w \arrow[r] \arrow[d] \arrow[dr, phantom, "\substack{\text{Additivity}\\ + h_J \\ \text{extension}}"] & v_\bullet\bar{w}_\bullet S_\bullet\mathcal{C}^w \arrow[r, "\simeq"', "L3 + L1"] \arrow[d] & v_\bullet w_\bullet S_\bullet\mathcal{C}^w \arrow[r, "\simeq\ \checkmark", "L2"'] \arrow[d] & wS_\bullet\mathcal{C}^w \arrow[d] \\
vS_\bullet\mathcal{C} \arrow[r] & v_\bullet\bar{w}_\bullet S_\bullet\mathcal{C} \arrow[r, "\simeq", "L3 + L1"'] & v_\bullet w_\bullet S_\bullet\mathcal{C} \arrow[r, "\simeq\ \checkmark", "L2"'] & wS_\bullet\mathcal{C}
\end{tikzcd}$$

cylinder functor / axiom + saturation

L2 = Swallowing lemma

of simplicial bicategories where

$$v_\bullet w_\bullet S_n\mathcal{C} = (vS_n\mathcal{C})(wS_n\mathcal{C}) \text{ for all } n \geq 0$$
$$\cup\!\!\mid$$
$$v_\bullet \bar{w}_\bullet S_n\mathcal{C} \quad \text{s.t. horizontal morphisms } v_\bullet \bar{w}_q S_n\mathcal{C} \text{ are also cofibrations.}$$

$$(vS_n\mathcal{C})_{p,q} = N_p vS_n\mathcal{C} \quad \forall q \geq 0$$

$$(wS_n\mathcal{C})_{p,q} = N_p wS_n\mathcal{C} \quad \forall q \geq 0$$

This implies

$$|vS_\bullet\mathcal{C}^w| \longrightarrow |vS_\bullet\mathcal{C}| \longrightarrow |wS_\bullet\mathcal{C}|$$

is a fiber sequence as desired.

Lemma 3 Let $(\mathscr{C}, c\mathscr{C}, w\mathscr{C})$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom such that $w\mathscr{C}$ satisfies the saturation axiom. Then the inclusion

$$i_*: |N_\cdot \bar{w}\mathscr{C}| \xrightarrow{\simeq} |N_\cdot w\mathscr{C}|$$

induces a homotopy equivalence.

Pf. Let $i: \bar{w}\mathscr{C} \to w\mathscr{C}$ denote the inclusion. By Quillen's theorem A it suffices to show

$$N_\cdot\, i/_B \simeq * \quad \text{for all } B \text{ in } w\mathscr{C}.$$

An object in i/B is a pair $(A; f: A \to B \in w_1\mathscr{C}$.

Since the cylinder functor satisfies the cylinder axiom $T(f) \xrightarrow{p(f)} B \in w_1\mathscr{C}$. We define a functor

$$\tilde{T}: i/_B \to i/B$$
$$(A, f: A \to B) \longmapsto (T(f), T(f) \xrightarrow{p(f)} B)$$

then $j_1(f), j_2(f) \in \bar{w}\mathscr{C}$ by the saturation axiom

So we have nat. trans.

$$\begin{array}{ccccc} (A, f: A \to B) & & (A, f: A \to B) & & (A, f: A \to B) \\ \downarrow \mathrm{id}; 1_B & \Rightarrow & \downarrow & \Rightarrow & \downarrow \\ (A, f: A \to B) & & (T(A), p(f): T(f) \to B) & & (A, id_B: B \to B) \end{array}$$

induced by

$$\left(A \xrightarrow{j_1} T(A) \xrightarrow{j_2} A \quad , \quad \begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[r, "j_2"] \arrow[d, "p"] & B \arrow[dl, equal] \\ & B & \end{tikzcd} \right)$$

So

$$id_{i/B} \simeq \tilde{f} \simeq const_B$$

$$\Rightarrow i/B \simeq * \text{ for all } B \text{ in } w\mathcal{C}. \quad \square$$

Consequently,

$$|v_\bullet \bar{w}_\bullet S_\bullet \mathcal{C}| \xrightarrow{\simeq} |v_\bullet w_\bullet S_\bullet \mathcal{C}|$$

and

$$|v.\bar{w}.S.\mathcal{C}| \xrightarrow{\simeq} |v.\bar{w}.S.\mathcal{C}|.$$

To prove the fibration theorem, it therefore suffices to show

$$\begin{tikzcd} vS.\mathcal{C}^w \arrow[r] \arrow[d] & v\bar{w}.S.\mathcal{C}^w \arrow[d] \\ vS.\mathcal{C} \arrow[r] & v\bar{w}.S.\mathcal{C} \end{tikzcd}$$

is a homotopy pullback and $|wS.\mathcal{C}^w| \simeq *$.

Since $w\mathscr{C}^{\omega}$ has an initial object,

$$|N_\bullet w\mathscr{C}^{\omega}| \simeq *.$$

Also, by the additivity theorem we saw that there is a homotopy fiber sequence

$$|N_\bullet vS_\bullet \mathscr{C}| \longrightarrow |N_\bullet vS^{(2)}_\bullet(f:\mathscr{C}^{\omega}\to\mathscr{C})| \to |N_\bullet vS^{(2)}_\bullet \mathscr{C}^{\omega}|$$

and a htpy equiv.

$$\to |N_\bullet vS^{(2)}_\bullet \mathscr{C}^{\omega}| \simeq |N_\bullet vS_\bullet \mathscr{C}^{\omega}|$$

so after rotating there is a homotopy fiber sequence

$$|N_\bullet vS_\bullet \mathscr{C}^{\omega}| \to |N_\bullet vS_\bullet \mathscr{C}| \to |N_\bullet vS^{(2)}_\bullet(f:\mathscr{C}^{\omega}\to\mathscr{C})|.$$

It therefore suffices to show

$$|v_\bullet \overline{w}_\bullet S_\bullet \mathscr{C}| \xrightarrow{\sim} |vS^{(2)}_\bullet(f:\mathscr{C}^{\omega}\to\mathscr{C})|$$

(where we regard $vS_\bullet(f:\mathscr{C}^{\omega}\to\mathscr{C})$ as a bisimplicial bicategory).

First, we show there is an equivalence of categories

$$w\mathcal{C} \quad \overline{w}_n\mathcal{C} \rightleftarrows S_n(f: \mathcal{C}^w \to \mathcal{C})$$

$$(A_0 \overset{\simeq}{\rightarrowtail} \cdots \overset{\simeq}{\rightarrowtail} A_n) \mapsto (A_1/A_0 \overset{\simeq}{\rightarrowtail} \cdots \overset{\simeq}{\rightarrowtail} A_n/A_0,\ A_0 \overset{\simeq}{\rightarrowtail} A_1 \overset{\simeq}{\rightarrowtail} \cdots \rightarrowtail A_n)$$

$\in w\mathcal{C}$ $\in w\mathcal{C}$

$$(B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n) \longleftarrow (B_1' \rightarrowtail \cdots \rightarrowtail B_n',\ B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n$$

s.t.

$$\begin{tikzcd}
B_0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B_1 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & B_n \arrow[d] \\
0 \arrow[r, tail] & B_1/B_0 \cong B_1' & & B_n/B_0 \cong B_n'
\end{tikzcd}$$

$\cong \in w\mathcal{C}$

Note: By the extension axiom

$$\begin{tikzcd}
B_0 \arrow[r, tail, "f"] & B_1 \arrow[r, tail] & B_1 \\
B_0 \arrow[u, "\mathrm{id}", "w\mathcal{C} \ni"'] \arrow[r, tail, "\mathrm{id}"'] & B_0 \arrow[u, "f"] \arrow[r, tail] & 0 \arrow[u, "\in w\mathcal{C}"']
\end{tikzcd}$$

$$\Rightarrow B_0 \overset{f}{\rightarrowtail} B_1 \in w\mathcal{C}.$$

Applying $v_\bullet S$, we get a map

$$v_\bullet S_\bullet(\overline{w}_\bullet \mathcal{C}) \longrightarrow v_\bullet S_\bullet(S_\bullet(f: \mathcal{C}^w \to \mathcal{C}))$$

s.t.

$$|v_p S_n(\overline{w}_\bullet \mathcal{C})| \xrightarrow{\simeq} |v_p S_n(S_\bullet(f: \mathcal{C}^w \to \mathcal{C}))|$$

is a htpy equivalence $\forall p, n$

so since

$$v_\bullet S_\bullet \overline{w}_\bullet \mathcal{C} = v_\bullet \overline{w}_\bullet S_\bullet \mathcal{C}$$

we have

$$|v_\bullet \overline{w}_\bullet S_\bullet \mathcal{C}| \xrightarrow{\simeq} |v_\bullet S_\bullet^{(2)}(f: \mathcal{C}^w \to \mathcal{C})|.$$

This finishes the proof.

Thm [Gillet–Waldhausen]

Let $\mathcal{C}$ be an exact category with $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{A}$ an abelian category such that $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$. Then the exact inclusion functor

$$\mathcal{C} \hookrightarrow Ch^b(\mathcal{C})$$

induces a homotopy equivalence

$$K(\mathcal{C}) \xrightarrow{\simeq} K(Ch^b(\mathcal{C}))$$

In particular,

$$K_n(\mathcal{C}) \cong K_n(Ch^b(\mathcal{C})) \quad \text{for all } n \geq 0.$$

Def. We say

$$0 \to A_n \to A_{n-1} \to \cdots \to A_0 \to 0$$

is an admissibly exact sequence in $\mathcal{C}$ if each map decomposes as

$$A_{n+1} \twoheadrightarrow B_n \rightarrowtail A_n \quad \text{such that}$$

$0 \to B_n \rightarrowtail A_n \twoheadrightarrow B_{n+1} \to 0$ is an exact sequence in $\mathcal{C}$ for all $n \geq 0$.

<u>Def</u>: Let $\mathcal{C}^{[a,b]}_{exact}$ be Waldhausen category w/ cofibrations the level-wise admissible monomorphisms $A_i \rightarrowtail A_i'$ such that

$$A_i \amalg_{B_i} B_i' \rightarrowtail A_i'$$

is an admissible monomorphism for each i and let weak equivalences be levelwise isomorphisms in $\mathcal{C}$. Then by the additivity theorem,

we can show

$$K(\mathcal{C}^{[a,b]}_{exact}) \simeq \prod_{k=a+1}^{b} K(\mathcal{C}).$$

<u>Lemma</u>: Consider the full subcategory $Ch^{[a,b]}(\mathcal{C})$ of $Ch^b(\mathcal{C})$ of those chain complexes $C_\bullet$ such that $C_i = 0$ when $i \notin [a,b]$. Then by the additivity theorem there is a homotopy equivalence

$$K(Ch^{[a,b]}(\mathcal{C})) \simeq \prod_{i=a}^{b} K(\mathcal{C}).$$

<u>Pf</u>: Exercise.

Proof of GW theorem.

level-wise admissible monos ↓ quasi-isos ↓

Consider the sequence

$$Ch^b(\mathcal{C})^{\omega} \simeq (Ch^b(\mathcal{C}), cCh^b\mathcal{C}, \mathrm{iso}Ch^b\mathcal{C}) \hookrightarrow (Ch^b(\mathcal{C}), cCh^b\mathcal{C}, wCh^b\mathcal{C})$$

$\downarrow$

A chain complexes that are quasi-isomorphic to 0

A.K.A. $\operatorname{colim}_n \mathcal{C}^{[-n,n]}_{exact}$

$\parallel$

A.K.A. $\operatorname{colim}_n Ch^{[-n,n]}(\mathcal{C})$

Recall: $K(\operatorname{colim}_n \mathcal{C}_n) \simeq \operatorname{colim}_n K(\mathcal{C}_n)$ (filtered colimit)

There are canonical fiber sequences

$$\begin{tikzcd} K(\mathcal{C}^{[-n,n]}_{exact}) \arrow[r] \arrow[d, phantom, "\wr"] & K(Ch^{[-n,n]}(\mathcal{C})) \arrow[r, "\chi"] \arrow[d, phantom, "\wr"] & K(\mathcal{C}) \\ \prod_{i=-n+1}^{n} K(\mathcal{C}) & \prod_{i=-n}^{n} K(\mathcal{C}) & \end{tikzcd}$$

for all n. Passing to colimits we have

$$\begin{tikzcd} \operatorname{colim}_n K(\mathcal{C}^{[-n,n]}_{exact}) \arrow[r] \arrow[d, phantom, "\wr"] & \operatorname{colim}_n K(Ch^{[-n,n]}(\mathcal{C})) \arrow[r, "\chi"] \arrow[d, phantom, "\wr"] & K(\mathcal{C}) \arrow[d] \\ K(Ch^b(\mathcal{C})^{\omega}) \arrow[r] & K(Ch^b(\mathcal{C}), \mathrm{iso}) \arrow[r] & K(Ch^b(\mathcal{C})) \end{tikzcd}$$

$$\Longrightarrow K(\mathcal{C}) \xrightarrow{\simeq} K(Ch^b(\mathcal{C})).$$