Lecture 8:

Waldhausen K-theory

I Waldhausen Categories

Recall: An exact category $\mathcal{A}$ is an additive category equiped with a fully faithful exact functor $\mathcal{A} \hookrightarrow \mathcal{B}$ where $\mathcal{B}$ is an abelian category. By a the Quillen-Gabriel embedding theorem we can equivalently define an exact category to be a pair (A, E) where A is an additive category and E is a class of an exact sequences

$$0 \to A \rightarrowtail B \twoheadrightarrow C \to 0 \quad \in E$$

satisfying

(1) E is closed under isomorphisms

(2) admissible monomorphisms are closed under composition & base change

(3) If a morphism $M \to M'$ has a kernel in $\mathcal{A}$ and $N \to M \to M'$ is an admissible epimorphism, then $M \to M'$ is an admissible epimorphism. The dual statement for admissible monos. holds

In fact, we will consider a generalization.

Def: A category with cofibrations consists of a pair $(\mathcal{C}, c\mathcal{C})$ where $\mathcal{C}$ is a category with a zero object 0 and $c\mathcal{C}$ is a subcategory satisfying

1) $c\mathcal{C}$ contains all isomorphisms (so $\mathrm{ob}\, c\mathcal{C} = \mathrm{ob}\, \mathcal{C}$)

2) the unique map $0 \rightarrowtail c$ is in $c\mathcal{C}$ for all c in $\mathcal{C}$.

3) if $f: A \rightarrowtail B$ is in $c\mathcal{C}$ and $A \to C$ is any morphism then the pushout

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r, tail] & C \amalg_A B \end{tikzcd}$$

exists in $\mathcal{C}$ and $C \rightarrowtail C \amalg_A B$ is in $c\mathcal{C}$

(Consequently, we have a notion of exact sequence

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ 0 \arrow[r, tail] & 0 \amalg_A B =: B/A. \end{tikzcd}$$)

<u>Def</u>: A **Waldhausen category** is a triple $(\mathscr{C}, c\mathscr{C}, w\mathscr{C})$ where $(\mathscr{C}, c\mathscr{C})$ is a category with cofibrations and $w\mathscr{C} \subseteq \mathscr{C}$ is a subcategory satisfying

1) all isomorphisms are in $w\mathscr{C}$
   (so $\mathrm{ob}\, w\mathscr{C} = \mathrm{ob}\, \mathscr{C}$)

2) $w\mathscr{C}$ satisfies the **gluing lemma**:
   Given a commutative diagram

$$\begin{tikzcd} D \arrow[d, "\wr"] & A \arrow[l] \arrow[r, tail] \arrow[d, "\wr"] & B \arrow[d, "\wr"] \\ D' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd}$$

where $D \xrightarrow{\sim} D'$, $A \xrightarrow{\sim} A'$, and $B \xrightarrow{\sim} B'$ are in $w\mathscr{C}$ and $A \rightarrowtail B$ and $A' \rightarrowtail B'$ are in $c\mathscr{C}$, then the induced map

$$D \amalg_A B \xrightarrow{\sim} D' \amalg_{A'} B'.$$

Example: Any exact category with

cofibrations = admissible monomorphisms

weak equivalences = isomorphisms

Example: Let $R^f_x \subseteq {}_{x\backslash}\mathrm{Top}/x$

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[rr, bend right, "id_X"'] & Y \arrow[r, "r"] & X \end{tikzcd}$$ such that (Y,X) is a relative CW complex with finitely many cells.

Then $(R_X, \mathrm{cof}\, R_X, wR_X)$ is a Waldhausen category where cofibrations are inclusions of sub CW complexes and weak equivalence(s) are weak equivalences after forgetting to Top.

A functor $\mathcal{C} \to \mathcal{D}$ between Waldhausen categories is exact if it preserves 0, cofibrations, weak equivalences, and pushouts

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ C \arrow[r, hook] & C \amalg_A B \end{tikzcd}.$$

Let Wald be the category of small Waldhausen categories and exact functors. We will define a functor $K : \mathrm{Wald} \longrightarrow \mathrm{CGWH}$

Construction: (Waldhausen's $S_\bullet$-construction)

Recall that Cat is the category of small categories and there is a fully faithful embedding

$$\Delta \hookrightarrow \mathrm{Cat}$$
$$[n] \longmapsto 0 \to 1 \to \dots \to n =: [n]$$

we write $[n]$ for the image of $[n]$ by abuse of notation. This forms a cosimplicial category with coface maps δ_i and codegeneracies σ_i. Let $\mathrm{Cat}(\mathscr{C}, D)$ denote the category of functors from $\mathscr{C}$ to D in Cat. Let $\mathrm{Arr}\,\mathscr{C} := \mathrm{Cat}([1], \mathscr{C})$.
Let $\mathscr{C}$ be in Wald and define

$$S_n \mathscr{C} \subseteq \mathrm{Cat}(\mathrm{Arr}([n]), \mathscr{C})$$

to be the full subcategory with objects

$A: \mathrm{Arr}([n]) \to \mathscr{C}$ such that

1) for every $\mu: [0] \to [n]$

$$A(\mu \circ \sigma_0) = 0$$

Ex:
$$\begin{tikzcd} A_{00} = 0 \arrow[r] \arrow[d] & A_{01} \arrow[d] \\ A_{01} = 0 \arrow[r] & A_{11} = 0 \end{tikzcd}$$

2) for every $\gamma: [2] \to [n]$, the sequence

$$A(\gamma \circ \delta_2) \rightarrowtail A(\gamma \circ \delta_1) \to A(\gamma \circ \delta_0) \text{ is}$$

is a cofibration sequence

Ex:
$$\begin{tikzcd} A_{00} = 0 \arrow[r] \arrow[d] & A_{01} \arrow[r, tail] \arrow[d] & A_{02} \arrow[d] \\ A_{01} = 0 \arrow[r] & A_{11} = 0 \arrow[r] & A_{12} \end{tikzcd}$$

$$\left(A(\gamma \circ \delta_0) \cong A(\gamma \circ \delta_1) / A(\gamma \circ \delta_2) \right).$$

Note: This produces a functor

$$S_n \mathcal{C} = \mathrm{Cat}(\mathrm{Cat}([1],[n]), \mathcal{C}) : \Delta^{op} \to \mathrm{Wald}$$

by letting $A' \rightarrowtail A$ be a cofibration

if for every functor $\theta : [1] \to [n]$

$A(\theta) \to A'(\theta)$ is an objectwise

cofibration.

A morphism $A \to A'$ is a weak

equivalence if for all $\theta : [1] \to [n]$

the map

$A(\theta) \to A(\theta)$ is a weak

equivalence objectwise.

The face and degeneracy maps have

an intuitive description as well, so

we will spell them out in

a special case.

Ex: Note that $S_0\mathcal{C} = \{0\}$, $S_1\mathcal{C} = \mathcal{C}$, and

$$S_2\mathcal{C} = \{A \rightarrowtail B \twoheadrightarrow B/A\}.$$

Consider an object in $S_3\mathcal{C}$

$$\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{03} \arrow[d, two heads] \\
& A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\
& & A_{23}
\end{tikzcd}$$

The maps $d_i : S_2\mathcal{C} \to S_1\mathcal{C}$ $0 \le i \le 3$ are defined as

$$d_0\left(\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{03} \arrow[d, two heads] \\
& A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\
& & A_{23}
\end{tikzcd}\right) = \begin{tikzcd}
A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\
& A_{23}
\end{tikzcd}$$

$$d_1\left(\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{03} \arrow[d, two heads] \\
& A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\
& & A_{23}
\end{tikzcd}\right) = \begin{tikzcd}
A_{02} \arrow[r, tail] & A_{03} \arrow[d, two heads] \\
& A_{23}
\end{tikzcd}$$

$$d_2\left(\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{03} \arrow[d, two heads] \\
& A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\
& & A_{23}
\end{tikzcd}\right) = \begin{tikzcd}
A_{01} \arrow[r, tail] & A_{03} \arrow[d, two heads] \\
& A_{13}
\end{tikzcd}$$

$$d_3\left(\begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{03} \arrow[d, two heads] \\
& A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\
& & A_{23}
\end{tikzcd}\right) = \begin{tikzcd}
A_{01} \arrow[r, tail] & A_{02} \arrow[d, two heads] \\
& A_{12}
\end{tikzcd}$$

We can consider the simplicial small category

$$\Delta^{op} \xrightarrow{wS_\bullet\mathcal{C}} Cat$$

by forgetting structure and postcompose with the nerve to produce

$$\Delta^{op} \xrightarrow{wS_\bullet\mathcal{C}} Cat \xrightarrow{N_\bullet} sSet$$

or in other words

$$N_\bullet wS_\bullet\mathcal{C}$$

is a bisimplicial set.

<u>Def</u>:

$$K^W(\mathcal{C}) := \Omega|N_\bullet wS_\bullet\mathcal{C}|$$

<u>Thm</u>: When $\mathcal{C} = P(R)$

$$\begin{array}{ccc} K_0(R) \times BGL(R)^+ & \simeq & K^W(P(R)) \\ \wr\!\wr & & \wr\!\wr \\ K^{\oplus}(P(R)) & \simeq & K^Q(P(R)) \end{array}$$

<u>Proposition</u> Let $\mathcal{C}$ be a small Waldhausen category, then

$$\pi_0 \Omega |N_\bullet wS_\bullet \mathcal{C}| \cong \mathbb{Z}[\mathrm{ob}\,\mathcal{C}] \Big/ \begin{array}{c} [C] \sim [C'] \\ [C] = [B] + [C/B] \end{array}$$

if $C \xrightarrow{\sim} C'$

if

$B \rightarrowtail C \twoheadrightarrow C/B$ cof. sequence

Proof:

$$\pi_0 \Omega |N_\bullet wS_\bullet \mathcal{C}|$$

$$\cong \pi_1 |N_\bullet wS_\bullet \mathcal{C}|$$

Since $N_0 wS_0 \mathcal{C} = \{0\}$

So $|N_\bullet wS_\bullet \mathcal{C}|$ is connected

Note: $|N_\bullet wS_\bullet \mathcal{C}| = |[n] \longmapsto BwS_n\mathcal{C}|$

Let $X_n := ([n] \longmapsto BwS_n\mathcal{C})$.

Then $|X_n| = |N_\bullet wS_\bullet \mathcal{C}|$.

$$\pi_1 |X_\bullet| = \pi_1 X_1 \Big/_{d_1 x = d_1 x \cdot d_0 x}$$

for $x \in \pi_0 X_2$

In our case,

$$\pi_0 X_2 = \pi_0 BwS_2\mathcal{C}$$
$$= \{[B \rightarrowtail C \twoheadrightarrow C/B]\}$$

(equiv. classes of cofibration sequences)

and

$$d_0\left(\begin{array}{c} B \rightarrowtail C \\ \quad\ \downarrow \\ \quad\ C/B \end{array}\right) = C/B$$

$$d_1\left(\begin{array}{c} B \rightarrowtail C \\ \quad\ \downarrow \\ \quad\ C/B \end{array}\right) = C$$

$$d_2\left(\begin{array}{c} B \rightarrowtail C \\ \quad\ \downarrow \\ \quad\ C/B \end{array}\right) = B$$

and since $\pi_0 X_1 = \text{ob}\,\mathcal{C}$

$$\pi_1 |X_\bullet| = \overbrace{F(\text{ob}\,\mathcal{C})}^{\text{free group}} \Big/ \begin{array}{c}[B] = [C] + [B/C] \\ \forall\ B \rightarrowtail C \twoheadrightarrow B/C\end{array} \cong \mathbb{Z}[\text{ob}\,\mathcal{C}] \Big/ [B] = [C] + [B/C]$$

Cor: $\pi_0 K^{w}(P(R)) \cong K_0(R)$

Basic properties

1) product preserving:

$$K(\mathcal{C} \times \mathcal{D}) \simeq K(\mathcal{C}) \times K(\mathcal{D})$$

2) homotopical:

Given exact functors

$$f, g : \mathcal{C} \longrightarrow \mathcal{D}$$

such that there is a natural transformation

$$T : f \Rightarrow g$$

such that $T_c : f(c) \to g(c)$ is a weak equivalence in $\mathcal{D}$ for all c in $\mathcal{C}$, then

$$K(f) \simeq K(g).$$

II The additivity theorem

Note: $|N_\bullet wS_\bullet \mathcal{C}| = \coprod_{n \geq 0} BwS_n\mathcal{C} \times |\Delta^n| \big/ \sim$

So we have a map

$$\begin{array}{ccc} sk_1 |N_\bullet wS_\bullet \mathcal{C}| & \longrightarrow & |N_\bullet wS_\bullet \mathcal{C}| \\ \| & & \\ \coprod_{k=0}^{1} BwS_k\mathcal{C} \times |\Delta^1| \big/ \sim & & \\ \cong & & \\ Bw\mathcal{C} \wedge S^1 & & \end{array}$$

So by adjunction, we produce a map

$$Bw\mathcal{C} \to \Omega |N_\bullet wS_\bullet \mathcal{C}|.$$

Also, since $S_\bullet \mathcal{C} : \Delta^{op} \to \mathrm{Wald}$ where $\mathcal{C}$ is a Waldhausen category, we can iterate it to form

$$S_\bullet^{(n)}\mathcal{C} = \underbrace{S_\bullet(\ \dots \ (}_{n} S_\bullet \mathcal{C}) \dots)$$

and this same construction produces a sequence

$$\begin{array}{ccccccc} Bw\mathcal{C} \to \Omega|N_\bullet wS_\bullet \mathcal{C}| & \xrightarrow{\sigma_0} & \Omega^2 |N_\bullet wS_\bullet^{(2)} \mathcal{C}| & \xrightarrow{\Omega\sigma_1} & \Omega^3 |N_\bullet wS_\bullet^{(2)} \mathcal{C}| & \to & \dots \\ \mathrel{\dot{=}\!\dot{=}} & & \mathrel{\dot{=}\!\dot{=}} & & \mathrel{\dot{=}\!\dot{=}} & & \\ K(\mathcal{C})_0 & & \Omega K(\mathcal{C})_1 & & \Omega^2 K(\mathcal{C})_2 & & \end{array}$$

<u>Thm</u> (Ω-Spectrum) The sequential spectrum

$$\{K(\mathcal{C})_n, \ \sigma_n : K(\mathcal{C})_n \to \Omega K(\mathcal{C})_{n+1}\}$$

is an Ω-spectrum; i.e the map

$\sigma_n : K(\mathcal{C})_n \to \Omega K(\mathcal{C})_{n+1}$ is a homeomorphism for all n.

Proof: Next time

<u>Thm</u> (Additivity)

1) The map $((d_1)_\bullet, (d_2)_\bullet) : K(S_2\mathcal{C}) \to K(\mathcal{C}) \times K(\mathcal{C})$ is a homotopy equivalence.

2) Given a sequence $F' \to F \to F'' : \mathcal{B}' \to \mathcal{C}$ of exact functors and natural transformations such that $F'(c) \rightarrowtail F(c) \twoheadrightarrow F''(c)$ is a cofibration sequence in $\mathcal{C}$ for all c in $\mathcal{B}'$, then

$$F_\bullet \simeq F'_\bullet \vee F''_\bullet$$

as maps of H-spaces.

First, there is an intermediate statement.

$$(*)\quad (d_0)_* + (d_2)_* = (d_1)_* : K(S_2\mathcal{C}) \to K(\mathcal{C}).$$

<u>Proof that $(*) \Rightarrow (2)$</u>

Specifying an exact sequence

$$F' \rightarrowtail F \twoheadrightarrow F'' : \mathcal{C}' \longrightarrow \mathcal{C}$$

is equivalent to specifying a functor

$$G : \mathcal{C}' \to S_2\mathcal{C}$$

such that $d_0 G = F'$, $d_1 G = F$, and $d_2 G = F''$. So

$$\begin{aligned} F_* = (d_1 G)_* &= (d_1)_* \circ (G_*) \\ &= ((d_0)_* \vee (d_2)_*) \circ G_* \\ &= (F')_* \vee (F'')_* . \end{aligned}$$

(2) $\Rightarrow$ (3)

After pre-composing with

$$|wS_\bullet \mathscr{C}| \times |wS_\bullet \mathscr{C}| \xrightarrow{V} |wS_\bullet S_2 \mathscr{C}|$$

$$A, B \longmapsto A \to A \vee B \to B$$

it is clear that

$$(d_1)_\otimes = (d_0)_\bullet V (d_1)_\otimes .$$

We will show V is a homotopy equivalence. The map V is clearly a section of

$$(A', B') \longleftarrow A' \to C' \to B'$$

$$|wS_\bullet \mathscr{C}| \times |wS_\bullet \mathscr{C}| \underset{V = s}{\overset{r = (d_{0\bullet}, d_{1\bullet})}{\rightleftarrows}} |wS_\bullet S_2 \mathscr{C}|;$$

$$(A, B) \longmapsto A \to A \vee B \to B$$

i.e. $r \circ s = id$ so s is a homotopy equivalence if r is.

Again, to prove (2) $\Rightarrow$ (1) we pass through an intermediate step.

<u>Def</u>: Let $A, B, \mathcal{C}$ be categories with cofibrations with A and B exact subcategories. We define $E(A, \mathcal{C}, B)$ to be the pullback

$$\begin{tikzcd} E(A, \mathcal{C}, B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S_2 \mathcal{C} \arrow[d] \\ A \times B \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

.

So objects in $E(A, \mathcal{C}, B)$ are exact sequences

$$A \rightarrowtail C \twoheadrightarrow B$$

where A is in the essential image of A and B is in the essential image of B.

We will show

$(2) \Rightarrow (**) \Rightarrow (1)$ where

$(**)\ ((d_0)_0, (d_1)_0) : K(\Sigma(A, \mathcal{C}, B)) \xrightarrow{\simeq} K(A) \times K(B)$

Note that $(**) \Rightarrow (1)$ is obvious, so we just need to show $(2) \Rightarrow (**)$.

Again, the map $r = ((d_0)_0, (d_1)_0)$ is a retract with section

$$s : K(A) \times K(B) \to K(\Sigma(A, \mathcal{C}, B))$$
$$A, B \longmapsto A \to A \vee B \to B$$

It therefore suffices to show

$$s \circ r \simeq id_{K(\Sigma(A, \mathcal{C}, B))}.$$

Consider the exact sequence of exact functors (18)

$$F' \rightarrowtail F \twoheadrightarrow F'' : \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$$

$$F'(A \rightarrowtail C \twoheadrightarrow B) = A \overset{=}{\rightarrowtail} A \twoheadrightarrow *$$

$$F(A \rightarrowtail C \twoheadrightarrow B) = A \rightarrowtail C \twoheadrightarrow B$$

$$F''(A \rightarrowtail C \twoheadrightarrow B) = * \rightarrowtail B \overset{=}{\twoheadrightarrow} B$$

Then $F'_* \vee F''_* \simeq F_*$ by (2).

and

$$(F'_* \vee F''_*)(A \rightarrowtail B \twoheadrightarrow C)$$

$$= A \rightarrowtail A \vee B \to C$$

$$= \mathrm{sor}(A \rightarrowtail B \to C)$$

So

$$\mathrm{sor} \simeq \mathrm{id}_{K(\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}))}$$

<u>Prop</u>: The additivity theorem holds iff $\{K(\mathscr{C})_n, \sigma_n\}$ is an Λ-spectrum

<u>Proof</u>: First, we observe that

$$|wS_2\mathscr{C}| \underset{(d_0)_* \vee (d_2)_*}{\overset{(d_1)_*}{\rightrightarrows}} |w\mathscr{C}| \xrightarrow{j} \Lambda|wS_\bullet\mathscr{C}|$$

then $(d_1)_* \circ j \simeq (d_0)_* \vee (d_2)_* \circ j$

by construction of the map j.

We have a map

$$|wS_2\mathscr{C}| \times |\Delta^2| \to \underline{\coprod_{i=0}^{2} |wS_i\mathscr{C}| \times |\Delta^i|}_{\sim} = |wS_\bullet\mathscr{C}|_{(2)}$$

where $|wS_\bullet\mathscr{C}|_{(2)}$ is the 2-skeleton of $|wS_\bullet\mathscr{C}|$.

Then we have a map

$$|wS_2\mathcal{C}| \xrightarrow{(d_1)_*} |w\mathcal{C}| \xrightarrow{j} \Omega|wS_\bullet\mathcal{C}|_{(2)}$$

and $|\Delta^2|$ gives a homotopy

from $(d_1)_* \circ j$ to

$$(d_{0*} \circ j) * (d_{2*} \circ j)$$

where $*$ is the concatenation of loops

which is homotopic to the

operation on the H-space $\Omega|wS_\bullet\mathcal{C}|_{(2)}$.

More generally,

$$|wS^{(n)}_\bullet(S_2\mathcal{C})| \underset{(d_0)_\bullet \vee (d_1)_*}{\overset{(d_1)_*}{\rightrightarrows}} |wS^{(n)}_\bullet\mathcal{C}| \xrightarrow{f} \Omega|wS^{(n+1)}_\bullet\mathcal{C}|$$

$$(d_1)_* \circ f \simeq (d_0)_* \vee (d_1)_* \circ f$$

Consequently, if

$$|wS_\bullet \mathcal{C}| \to \lambda |wS_\bullet^{(2)} \mathcal{C}|$$

is a homeomorphism then

$$|wS_\bullet (S_2 \mathcal{C})| \underset{(d_0)_\bullet \vee (d_2)_\bullet}{\overset{(d_1)_\bullet}{\rightrightarrows}} |wS_\bullet \mathcal{C}| \xrightarrow{\cong} \lambda |wS_\bullet^{(2)} \mathcal{C}|$$

implies

$$(d_1)_\bullet \simeq (d_0)_\bullet \vee (d_2)_\bullet$$

$\Rightarrow$ additivity

Conversely, if the additivity theorem holds this implies

$$|wS_\bullet^{(n)} \mathcal{C}| \to \lambda |wS_\bullet^{(n+1)} \mathcal{C}|$$

is a homeomorphism.

we will defer the proof until later. Given a functor

$$X : J \to Top$$

define

$$\operatorname*{holim}_{k \in J} X_k := \left| [m] \mapsto \coprod_{j \in N_m J} X j_0 \right|$$

where $j = (j_0 \to j_1 \to \cdots \to j_m)$

Let

$$\Lambda^\infty_{naive} K(\mathcal{C}) := \operatorname*{hocolim}_{n \in \mathbb{N}} \Lambda^n |wS_\bullet^{(n)} \mathcal{C}|$$

Prop: The additivity theorem holds for

$\Lambda^\infty_{naive} K(\mathcal{C})$.

Proof The two composites

$$|wS_\bullet^{(n)} S_2 \mathcal{C}| \underset{(d_0)_* \vee (d_1)_*}{\overset{(d_1)_*}{\rightrightarrows}} |wS_\bullet^{(n)} \mathcal{C}| \xrightarrow{j} |wS_\bullet^{(n+1)} \mathcal{C}|$$

So after applying $\operatorname{hocolim}_{n \in \mathbb{N}}$ we have

$$\Omega^\infty_{\text{naive}} K(S_2 \mathcal{C}) \underset{(d_0)_* \vee (d_1)_*}{\overset{(d_1)_*}{\rightrightarrows}} \Omega^\infty_{\text{naive}} K(\mathcal{C}) \xrightarrow{\cong} \Omega^\infty_{\text{naive}} K(\mathcal{C})$$

So the additivity theorem holds for $\Omega^\infty_{\text{naive}} K(\mathcal{C})$. //

This leads us to the following.

<u>Def</u>: A global Euler characteristic is a pair (E, χ) where

$E : \mathrm{Wald} \longrightarrow \mathrm{CGWH}$ is a functor

and $\chi : \mathrm{ob}(-) \longrightarrow E(-)$ is a natural transformation such that

1) $E(\mathcal{C} \times \mathcal{D}) \xrightarrow{\sim} E(\mathcal{C}) \times E(\mathcal{D})$

2) E satisfies the additivity theorem

3) The functor

$$\mathcal{C} \to w_1\mathcal{C} := \mathrm{Arr}(w\mathcal{C})$$
$$c \longmapsto c \xrightarrow{id} c$$

induces an equivalence

$$E(\mathcal{C}) \xrightarrow{\simeq} E(w_1\mathcal{C})$$

4) $E(\mathcal{C})$ is a group-like H-space.

<u>Rmk</u>: By a result of Segal

(1) $E(\mathscr{C}) \simeq \Omega^n Y_n$ for some space Y_n for all n; i.e.

$$\begin{tikzcd} E : \mathrm{Wald} \arrow[r] \arrow[dr] & \mathrm{CGWH}_* \\ & \text{infinite loop spaces} \arrow[u, hook] \end{tikzcd}$$

(2) If $c \rightarrowtail d \twoheadrightarrow e$ is a cofiber sequence in $\mathscr{C}$ then

$$\chi(c) + \chi(e) \simeq \chi(d)$$

where $+$ is the operation in the H-space $E(\mathscr{C})$, so χ is indeed an Euler characteristic.

To see this, note that

$$d_0, d_1, d_2 : S_2\mathcal{C} \longrightarrow \mathcal{C}$$

$$c \rightarrowtail d \twoheadrightarrow e \longmapsto c, d, e$$

satisfy

$$E(d) = E(c) + E(e)$$

and we have

$$\chi_{\mathcal{C}} : \mathrm{ob}\,\mathcal{C} \to E(\mathcal{C})$$

$$\chi_{\mathcal{C}}(c) = E(c) \quad \forall\, c \in \mathcal{C}$$

so

$$\chi_{\mathcal{C}}(d) = \chi_{\mathcal{C}}(c) + \chi_{\mathcal{C}}(e).$$

Def: We say

$$(A,\chi_A) \to (B,\chi_B)$$

is a weak equivalence if

$$A(\mathscr{C}) \xrightarrow{\simeq} B\mathscr{C} \text{ is a}$$

homotopy equivalence for all Waldhausen categories $\mathscr{C}$.

We write $ho(Eul)$ for the category whose objects are Euler characteristics and morphisms are

$$ho(Eul)(A,B) := Eul(A,B)/f \sim g$$

where $f \in Eul(A,B)$ if $A(f) \simeq B(g)$

if $f : A \Rightarrow B$ and $\chi_B \circ f = \chi_A$.

<u>Thm</u> (Universal property)

The pair (K, X_{univ}) is the initial object in

$$\mathrm{Ho}(\mathcal{E}ul).$$

Here

$(X_{univ})_{\mathscr{C}} : \mathrm{ob}\,\mathscr{C} \to K(\mathscr{C})$

is the adjoint of

$$\begin{array}{l} \mathrm{Sk}_1 K(\mathscr{C}) \longrightarrow |[n] \mapsto B_w S_n \mathscr{C}| \\ \quad \shortparallel \\ B_w S_1 \mathscr{C} \times \Delta^1 \\ \quad \cong\!\!\wr \\ \mathrm{ob}\,\mathscr{C} \wedge S^1 \end{array}$$

Proofs next time!