Loday construction in functor categories

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Motivation

Goal:

Describe a computational tool (using some categorical machinery) for computing higher order THH, which is an approximation to iterated algebraic K-theory.

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Algebraic K-theory $K(R)$	iterated algebraic K-theory $K^{(n)}(R)$)
vector bundles	n-vector bundles K(2-vector bundles $/k) \simeq K(K(k))$ (Baas-Dundas-Richter-Rognes, Osorno)
E_1-ring spectra	E_n-ring spectra Deligne conjecture for K-theory (Blumberg-Gepner-Tabuada, Barwick)
Quillen-Lichtenbaum	Ausoni-Rognes red-shift conjecture $K^{(n)}$: ht. m spectra $\to$ ht. $m+n$ spectra

Approximating iterated algebraic K-theory

For a commutative ring spectrum R, we can approximate iterated K-theory using the iterated trace map

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where $THH(R) = S^1 \otimes R$. Now we observe that

$$S^1 \otimes (S^1 \otimes \dots (S^1 \otimes R)) = T^n \otimes R$$

since Comm$\mathcal{S}$ has all colimits weighted in sSets (McClure-Staffeldt). We'd therefore like to have tools for computing $T^n \otimes R$ and related invariants.

The Loday construction

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Definition

Define $X_\bullet \tilde{\otimes} I$ to be the simplicial object in $\mathcal{C}$ with n-simplices $(X_\bullet \tilde{\otimes} I)_n = \bigotimes_{x \in X_n} I\{x\}$ and with face maps

$$d_i : \bigotimes_{x \in X_n} I\{x\} \to \bigotimes_{y \in X_{n-1}} I\{y\}$$

defined by

$$\bigotimes_{x \in X_n} I\{x\} \xrightarrow{\cong} \bigotimes_{y \in X_{n-1}} \bigotimes_{x \in d_i^{-1}(y)} I\{x\} \longrightarrow \bigotimes_{y \in X_{n-1}} I\{y\}$$

and similarly for degeneracy maps.

Functor categories: symmetric monoidal product

Let $(\mathcal{D}, \otimes)$ and $(\mathcal{C}, \wedge)$ be symmetric monoidal categories enriched in $\mathcal{C}$.

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$$(I \otimes_{Day} J)(c) = \int^{(a,b) \in \mathcal{D} \times \mathcal{D}} \mathcal{D}(a \otimes b, c) \wedge I(a) \wedge I(b)$$

and on morphisms by induced maps of coends.

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Theorem (Day)

The category $(\mathrm{Fun}(\mathcal{D}, \mathcal{C}), \otimes_{Day})$ is a closed symmetric monoidal category and the category of commutative monoids in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is equivalent to the subcategory category of lax symmetric monoidal functors $\mathrm{Fun}^{LS}(\mathcal{D}, \mathcal{C})$.

Functor categories: model structure

Let $\mathcal{C}$ be a combinatorial cofibrantly generated symmetric monoidal model category satisfying the SS monoid axiom and let $\mathcal{D}$ have virtually cofibrant function spaces.

Theorem (Isaacson)

The category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ with the projective model structure and Day convolution is a symmetric monoidal model category that satisfies the SS monoid axiom.

Following White, we say $\mathcal{M}$ satisfies the strong commutative monoid axiom (SCMA) if for any (acyclic) cofibration h the map $h^{\square n}/\Sigma_n$ is also an (acyclic) cofibration.

Functor categories: model structure

Lemma (A-K)

In addition, let $\mathcal{C}$ satisfy the SCMA, and let $\mathcal{D}$ be a POSet enriched in $\mathcal{C}$ ($\mathcal{D}(a, b) = 1_{\mathcal{C}}$ or 0). Then the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ with the projective model structure satisfies the strong commutative monoid axiom.

White proved that if a model category $\mathcal{M}$ satisfies (SCMA) then Comm$\mathcal{M}$ has the model structure inherited from $\mathcal{M}$ and cofibrations in Comm$\mathcal{M}$ with cofibrant source forget to cofibrations in $\mathcal{M}$. In particular, cofibrant objects in $\mathrm{Fun}^{LS}(\mathcal{D}, \mathcal{C})$ forget to cofibrant objects in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Filtered commutative ring spectra

Definition

A filtered commutative ring spectrum is a cofibrant object in $\mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$ with the model structure inherited from the projective model structure on $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$

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This is the same data as a sequence of cofibrations

$$\dots I(2) \to I(1) \to I(0)$$

between cofibrant objects with structure maps

$$\rho_{i,j} : I(i) \wedge I(j) \to I(i+j)$$

satisfying commutativity, associativity, unitality, and compatibility.

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satisfying commutativity, associativity, unitality, and compatibility. Now, for a simplicial (finite) set $X_\bullet$ we can form $X_\bullet \otimes I$ where $I \in \mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$ where $\mathcal{S}$ is the category of symmetric spectra of simplicial sets (with the positive flat stable model structure).

May filtration of the generalized bar construction

Let $I \in \mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$, then the May filtration of $X_\bullet \otimes I(0)$ is the Loday construction in $\mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$;

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$$\begin{array}{ccccccc}
(\bigotimes^{Day}_{x \in X_0} I)(0) & \leftleftarrows\!\!\rightarrow & (\bigotimes^{Day}_{x \in X_1} I)(0) & \overset{\leftarrow}{\underset{\leftarrow}{\rightrightarrows}} & (\bigotimes^{Day}_{x \in X_2} I)(0) & \cdots & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
(\bigotimes^{Day}_{x \in X_0} I)(1) & \leftleftarrows\!\!\rightarrow & (\bigotimes^{Day}_{x \in X_1} I)(1) & \overset{\leftarrow}{\underset{\leftarrow}{\rightrightarrows}} & (\bigotimes^{Day}_{x \in X_2} I)(1) & \cdots & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
(\bigotimes^{Day}_{x \in X_0} I)(2) & \leftleftarrows\!\!\rightarrow & (\bigotimes^{Day}_{x \in X_1} I)(2) & \overset{\leftarrow}{\underset{\leftarrow}{\rightrightarrows}} & (\bigotimes^{Day}_{x \in X_2} I)(2) & \cdots & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Example:

The object $(I \otimes_{\mathrm{Day}} I)(n)$ is the colimit of the n-th truncation of the diagram

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & I(2) \wedge I(2) & \longrightarrow & I(1) \wedge I(2) & \longrightarrow & I(0) \wedge I(2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & I(2) \wedge I(1) & \longrightarrow & I(1) \wedge I(1) & \longrightarrow & I(0) \wedge I(1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & I(2) \wedge I(0) & \longrightarrow & I(1) \wedge I(0) & \longrightarrow & I(0) \wedge I(0).
\end{array}
$$

THH-May spectral sequence

Theorem (A-K, Salch)

If I is a cofibrant object in $\mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$ then $|X_\bullet \otimes I|$ is again a cofibrant object in $\mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$.

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Given a cofibrant object I in $\mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$ we can form a commutative ring spectrum $E_0 I$, which is additively $E_0 I = \bigvee_{i \geq 0} I(i)/I(i+1)$.

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Theorem (A-K, Salch)

There is a spectral sequence

$$G_{*,*}(X_\bullet \otimes E_0 I) \Rightarrow G_*(X_\bullet \otimes I(0))$$

for any generalized homology theory G.

THH-May spectral sequence

The main step in constructing this spectral sequence is the proof that the construction of the associated graded E_0 commutes with the functor $X_\bullet \otimes -$.

Theorem (A-K, Salch)

There is an equivalence in Comm $\mathcal{S}$

$$E_0(X_\bullet \otimes I) \simeq X_\bullet \otimes E_0 I.$$

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Theorem (A-K, Salch)

There is an equivalence in $\mathrm{Comm}\,\mathcal{S}$

$$E_0(X_\bullet \otimes I) \simeq X_\bullet \otimes E_0 I.$$

(The object on the left is the associated graded object in $\mathrm{Comm}\,\mathcal{S}$ of the filtered commutative ring spectrum $X_\bullet \otimes I$. A priori, this is the E_1-page of the THH-May spectral sequence.)

Computation

Let $j = K(\mathbb{F}_q)_p$ where $p \geq 5$ and q is a prime power that topologically generates $\mathbb{Z}_p^\times$.

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Theorem (A-K)

There is an isomorphism of graded rings

$$V(1)_* THH(j) \cong P(\mu_2) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, \alpha_1, \lambda_1', \lambda_2\alpha_1, \lambda_2\lambda_1'\alpha_1\lambda_1'\lambda_2\}.$$

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Proof sketch.

Construct the Whitehead tower $j^{\geq \bullet}$ of j as an object in $\mathrm{Fun}^{LS}(\mathbb{N}^{\mathrm{op}}, \mathcal{S})$, then compute

$$V(1)_{*,*} THH(E_0 j^{\geq \bullet}) \Rightarrow V(1)_* THH(j).$$

□

Thank You

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