CHROMATIC COMPLEXITY OF THE ALGEBRAIC K-THEORY OF $y(n)$

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Abstract

The family of Thom spectra $y(n)$ interpolate between the sphere spectrum and the mod two Eilenberg-MacLane spectrum. Computations of Mahowald, Ravenel, and Shick show that the E_1-ring spectrum $y(n)$ has chromatic complexity n for $0 \leq n \leq \infty$. In this paper, we show that relative algebraic K-theory associated to the map $y(n) \to H\mathbb{F}_2$ has chromatic complexity at least $n+1$, up to a conjecture about the Morava K-theory of inverse limits. This gives evidence for a variant of the red-shift conjecture of Ausoni and Rognes at all chromatic heights.

Contents

1. Introduction

Algebraic K-theory encodes deep arithmetic properties of rings. For example, the low-dimensional algebraic K-theory groups recover the Picard group, the group of units, and the Brauer group. The higher algebraic K-groups also contain arithmetic data. For example, the Lichtenbaum-Quillen Conjecture [34, 45], which has been proven by Voevodsky [50] building on work of Rost (see [28] for survey), says that the ℓ-adic étale cohomology of certain rings of integers in number fields can be related to their algebraic K-theory. Consequently, algebraic K-groups encodes special values of Dedekind zeta functions. Some of the first evidence for the Lichtenbaum-Quillen conjecture came from Hesselholt-Madsen's computations of algebraic K-theory local fields [30]. Since these computations relied on understanding $K(\mathbb{F}_p)$, we consider $K(\mathbb{F}_p)$ to be of fundamental importance to arithmetic applications of algebraic K-theory.

On the other hand, algebraic K-theory encodes interesting geometric and topological properties of spaces. For example, Wall's finiteness obstruction and the Whitehead torsion of a space X are encoded in the first two algebraic K-groups of the group ring $\mathbb{Z}[\pi_1 X]$. Moreover, the higher algebraic K-theory of spaces, called A-theory, encodes geometric data related to surgery theory, pseudo-isotopies, and concordance [33]. Since a point is the initial object in spaces, the A-theory of any space admits a map from the A-theory of a point $A(*)$. As such, we consider $A(*)$ to be of fundamental importance to geometric applications of algebraic K-theory.

With modern definitions, algebraic K-theory can be defined for "brave new rings," or more precisely, E_1-ring spectra. The algebraic K-theory of E_1-ring spectra generalizes the previous examples as follows. For a ring R, the algebraic K-groups $K_n(R)$ can be recovered as the homotopy groups of the algebraic K-theory spectrum $K(HR)$ of the Eilenberg-MacLane spectrum for R. For a space X, the A-groups $A_n(X)$ can be recovered as the homotopy groups of the algebraic K-theory spectrum $K(\Sigma^\infty_+ \Omega X)$ of the suspension spectrum of ΩX. The fundamental examples above may therefore be recovered from $K(H\mathbb{F}_p)$ and $K(S)$, respectively.

Mahowald showed in [37] that $H\mathbb{F}_2$ can be constructed as the Thom spectrum of a two-fold loop map $\Omega^2 S^3 \to BGL_1 S$, i.e.

$$H\mathbb{F}_2 \simeq Th(\Omega^2 S^3 \to BGL_1 S)$$

where $BGL_1 S$ is a model for the classifying space of stable spherical bundles. Using the identification of $\Omega\Sigma S^2$ with the James construction $J_\infty S^2$, one can define spectra

$$y(n) := Th(\Omega J_{2^{n+1}-1} S^2 \to BGL_1 S)$$

by restricting the above map to a truncation of the James construction (see Section 2 for details). Functoriality of the Thom construction provides maps of Thom spectra

$$S \simeq y(0) \to y(1) \to y(2) \to \cdots \to y(\infty) \simeq H\mathbb{F}_2.$$

The spectra $y(n)$ interpolate between S and $H\mathbb{F}_2$, and thus the spectra $K(y(n))$ interpolate between $A(*)$ and $K(\mathbb{F}_2)$. The goal of this paper is to understand how certain computational techniques (e.g. homological trace methods) and chromatic invariants (e.g. type) vary as one interpolates between the arithmetic world where $K(\mathbb{F}_p)$ is fundamental and the geometric world where $K(S)$ is fundamental.

We say that a spectrum X has type n if $K(n)_*X \ncong 0$ and $K(i)_*X \cong 0$ for all $0 \le i \le n-1$. Usually this notion of chromatic height is used for finite spectra, in which case $K(n-1)_*X \cong 0$ implies $K(i)_*X \cong 0$ for $0 \le i < n-1$. In Mahowald-Ravenel-Shick they show that $y(n)$ has type n (see Lemma 2.16 for an alternate proof). The red-shift conjecture of Ausoni-Rognes [10] suggests that algebraic K-theory of a spectrum of chromatic complexity n has chromatic complexity $n+1$. Given a functor F valued in spectra, let $F(A,B) := fib(F(A) \to F(B))$ denote the *relative F of A with respect to B*. The main theorem of this paper gives evidence for a variant of the red-shift conjecture for relative algebraic K-theory at all chromatic heights.

Theorem (Theorem 6.10)**.** *The relative algebraic K-theory spectrum $K(y(n), H\mathbb{F}_p)$ has type at least $n+1$; i.e. there is an isomorphism*

$$K(m)_*K(y(n), H\mathbb{F}_p) \cong 0$$

for each natural number m such that $0 \le m \le n$.

Note that this theorem currently relies on Conjecture 6.1, which is considered by the first author and Salch in work in progress [5]. The main theorems proven in this paper which are not reliant on Conjecture 6.1 can be found at the end of Section 1.2.

1.1. **Trace methods.** Though algebraic K-theory is an important invariant with many applications, it is notoriously difficult to compute. For rings, Goodwillie [25] showed that rational information about algebraic K-theory can be computed using a nontrivial trace map from algebraic K-theory to Hochschild homology. This map factors through an invariant called negative cyclic homology, which is a close approximation to algebraic K-theory rationally by [25] .

In [18], Bökstedt defined topological Hochschild homology, which in a modern reformulation, can be constructed using the cyclic bar construction in the symmetric monoidal category of spectra. Bökstedt, Hsiang, and Madsen [20] then developed the theory of topological cyclic homology and constructed a trace map

$$K(R) \longrightarrow TC(R)$$

from the algebraic K-theory to topological cyclic homology. An important theorem of Dundas, Goodwillie, and McCarthy [22] shows that relative algebraic K-theory and relative topological cyclic homology agree for maps of connective ring spectra that induce a surjective map on π_0 with nilpotent kernel. Since topological cyclic homology is computable from topological Hochschild homology, this provides a method for computing algebraic K-theory known colloquially as trace methods.

Trace methods have led to several striking results in algebraic K-theory which have had fascinating arithmetic, geometric, and chromatic homotopy theoretic applications. Here we only highlight the major contributions most relevant to the present paper. On the arithmetic side, Hesselholt and Madsen demonstrated the power of trace methods in [29] by computing the algebraic K-theory of finite algebras over the Witt vectors of perfect fields. This computation played an important role in their verification of the Lichtenbaum-Quillen Conjecture for local fields [30]. On the geometric side, Bökstedt-Hsiang-Madsen computed the homotopy type of $TC(S)$ in [20]. Later Rognes [47] used their description to compute $TC(S)$ in a range and Blumberg and Mandell [15] extended these calculations. Finally, computations of Ausoni and Rognes [9] on mod (p, v_1) algebraic K-theory of the Adams summand also broke new ground on algebraic K-theory of ring spectra and led to further computations of algebraic K-theory of complex K-theory [8] and connective Morava K-theory [11]. These computations suggest a fascinating interaction between algebraic K-theory and chromatic homotopy theory.

More recently, there is renewed interest in topological Hochschild homology and related invariants because of its role in work of Bhatt, Morrow, and Scholze on integral p-adic Hodge theory [14]. This work relies on a new construction of topological cyclic homology due to Nikolaus and Scholze [43], which also makes topological cyclic homology more computationally accessible. Bhatt, Morrow, and Scholze [14] use topological Hochschild homology and the related invariants topological negative cyclic homology and topological periodic cyclic homology to construct a cohomology theory that recovers de Rham cohomology, crystalline cohomology, and étale cohomology.

1.2. **Chromatic homotopy theory.** Chromatic stable homotopy theory provides a stratification of the stable homotopy category by "chromatic complexity." The main result of this paper gives evidence for a variant of the red-shift conjecture of Ausoni-Rognes [10], which suggests that algebraic K-theory increases chromatic complexity by one. Here we make the case that this conjecture is interesting to study for many different notions of chromatic complexity, and therefore we briefly summarize some well known notions of chromatic complexity:

(1) If X is a finite complex, then by its chromatic complexity we mean its *type*, which is determined by applying the homology theory known as Morava K-theory with coefficients $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ for $n > 0$ whereas $K(0) = H\mathbb{Q}$. In particular, X is type n if $K(n-1)_*X \cong 0$, but $K(n)_*X \ncong 0$. For example, the chromatic complexity of the mod p Moore spectrum is one, because $K(1)_*S/p \ncong 0$, but S/p is rationally acyclic. The notion of type also makes sense when X is not a finite complex, but in that case we say that X has type n if $K(n)_*X \ncong 0$ and $K(i)_*X \cong 0$ for $0 \le i \le n-1$.
(2) If R is a $K(n)$-local (pro-)Galois extension of the $K(n)$-local sphere $L_{K(1)}S$ in the sense of Rognes [48] then we consider R to be chromatic complexity n. For example, the Adams summand of complex K-theory L is a $K(1)$-local $\mathbb{S}_1$-pro-Galois extension of the $K(1)$-local sphere, where $\mathbb{S}_1$ is the first Morava stabilizer group of automorphisms $\mathrm{Aut}(\Gamma_1)$ of the Honda formal group law of height one Γ_1 .
(3) The following definition appears in [13, Def. 6.1]. Let $\mathcal{T}_X$ be the thick subcategory of all finite p-local spectra V such that the map
$$V \wedge X \to v_n^{-1} V \wedge X$$
induces an equivalence on homotopy groups in sufficiently high degrees, then we say X has telescopic complexity n if $\mathcal{T}_X = \mathcal{C}_n$ where $\mathcal{C}_n$ is the thick subcategory of spectra of type n. By the thick subcategory theorem of Hopkins-Smith [32] we know that $\mathcal{T}_X = \mathcal{C}_m$ for some m, because $\mathcal{T}_X$ is thick, so this notion is well defined. Here we also use the periodicity theorem of [32], which says that every finite p-local spectrum has a v_n power self map v_n^k for some n

and k and we can then define $v_n^{-1}V$ as the telescope

$$v_n^{-1}V = \mathrm{colim}(V \xrightarrow{v_n^k} \Sigma^{-(2kp^n-2)}V \xrightarrow{v_n^k} \Sigma^{-(4kp^n-2)}V \xrightarrow{v_n^k} \ldots).$$

(4) We say a ring spectrum R has *detection height* n if n is the smallest natural number such that the n-th primary Greek letter family $\alpha_k^{(n)}$ is detected in the Hurewicz image of R. This notion is particularly well-suited to examples such as the connective cover of the $K(1)$-local sphere.

In order to cover several conjectures with the same umbrella, we present the following meta-conjecture version of the red-shift conjecture.

Meta-Conjecture 1.1. *If R is a ring spectrum with chromatic complexity n, then $K(R)$ is a spectrum with chromatic complexity $n+1$.*

The precise statement of the red-shift conjecture in [10] uses Item (2) to measure chromatic complexity of the input and Item (3) to measure chromatic complexity of the output.

Conjecture 1.2 (Ausoni-Rognes red-shift conjecture)**.** *If R is a* suitably finite $K(n)$-local spectrum*, then $K(R)$ has telescopic complexity $n+1$.*

Here, 'suitably finite' means that the Galois group in Item (2) is a finite group. Ausoni and Rognes showed that the algebraic K-theory $K(\ell)$ of the connective p-complete Adams summand of complex K-theory has telescopic complexity two and fp-type 2 in [9]. As a consequence of this result and the localization sequence

$$K(\mathbb{Z}) \to K(\ell) \to K(L)$$

of Blumberg and Mandell [17], one can show that $K(L)$ also has telescopic complexity 2, where L is the periodic Adams summand. In particular, we see a shift from chromatic complexity 1 to chromatic complexity 2. This provides evidence for Conjecture 1.2.

Work of the first author suggests a different approach to the Meta-Conjecture 1.2. The connective cover of the $K(1)$-local sphere $\tau_{\geq 0}L_{K(1)}S$ detects the α-family, which is a periodic family of height one in the stable homotopy groups of spheres. The first author showed that the primary β-family is detected in $K(\tau_{\geq 0}L_{K(1)}S)$ in [4]. This gives evidence for the following conjecture due to the first author [4], which is the version of Meta-conjecture 1.1 where chromatic complexity is measured using Item (4) for the input and output.

Conjecture 1.3 (Greek letter family red-shift conjecture)**.** *Let R be an E_1-ring spectrum and p a prime such that the primary n-th Greek letter family $\alpha_k^{(n)}$ and $n+1$-st primary Greek letter family $\alpha_k^{(n+1)}$ are nontrivial in the homotopy groups of spheres. If $\alpha_k^{(n)}$ is nontrivial in the Hurewicz image of R, then $\alpha_k^{(n+1)}$ is nontrivial in the Hurewicz image of $K(R)$.*

The calculations of Mahowald, Ravenel, and Shick [38] imply that for all $n \geq 0$, the spectrum $y(n)$ has chromatic complexity n and we provide our own proof in Lemma 2.16. In other words, $K(m)_*(y(n)) \cong 0$ for $m < n$ and $K(n)_*(y(n)) \ncong 0$. The main theorems of this paper concern continuous Morava K-theory of certain filtrations of topological periodic cyclic homology of $y(n)$ denoted $\{TP(y(n))[i]\}$ and topological negative cyclic homology of $y(n)$ denoted $\{TC^-(y(n))[i]\}$, which we define in Section 4 and Section 5 respectively.

Let F be any of the functors K, TC, TC^-, TP, $TC^-(-)[i]$, or $TP(-)[i]$ and let

$$F(A,B) := fib(F(A) \to F(B))$$

denote the *relative F of A with respect to B*. We combine the two main computational results in the following theorem

Theorem (Theorems 4.24, 5.6)**.** *There are isomorphisms*

$$K(m)^c_*TC^-(y(n)) \cong K(m)^c_*TP(y(n)) \cong 0.$$

for all $1 \leq m \leq n$.

The main result of this paper, Theorem 6.10, relies on Theorems 4.24, 5.6 as well as a synthesis of several recent developments in trace methods and some of the foundational results on algebraic K-theory. Theorems 4.24 and 5.6 which concern TP and TC^{-}, respectively, use the homological trace methods approach pioneered by Bruner and Rognes [21]. Combining these results with Conjecture 6.1, the subject of work in progress of the first author and Salch [5], implies Theorem 6.3. Combining Thereom 6.3 with the formula for topological cyclic homology due to Nikolaus-Scholze [43] proves Theorem 6.6. Then combining Thereom 6.6 with Quillen's computation of the algebraic K-theory of finite fields [44], Hesselholt-Madsen's identification of $TC(H\mathbb{F}_2)$ in [29], the Dundas-Goodwillie-McCarthy theorem [22], and work of Waldhausen [51] proves the main theorem, Theorem 6.10. Therefore, assuming Conjecture 6.1, the main theorem of this paper is evidence for Meta-Conjecture 1.2 in relative algebraic K-theory at all chromatic heights where chromatic complexity is measured using type for the input as well as the output.

1.3. **Outline.** In Section 2, we recall the construction and basic properties of the spectra $y(n)$. We then provide a proof that $y(n)$ has type n for all $n \geq 1$ using Margolis homology and the localized Adams spectral sequence. We also construct Thom spectra $z(n)$ which are integral analogs of $y(n)$, i.e. they are spectra which interpolate between S and $H\mathbb{Z}$. Again, using Margolis homology, we show that the spectra $z(n)$ have type n, that they have a v_n-self map, and that the cofiber of this v_n-self map is a spectrum of type $n+1$. We believe that the spectra $z(n)$ are of independent interest.

In Section 3, we analyze the Bökstedt spectral sequence converging to the mod two homology of $THH(y(n))$. We also a prove a key technical proposition (Proposition 3.12) about the map $H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$ which we use in subsequent sections.

In Section 4, we analyze the topological periodic cyclic homology of $y(n)$ defined as the Tate construction

$$TP(y(n)) := THH(y(n))^{t\,\mathbb{T}}$$

of $THH(y(n))$ with respect to the canonical circle action on topological Hochschild homology. We begin by explaining why the usual Tate spectral sequence cannot be computed to determine the homotopy groups $\pi_*(TP(y(n)))$. We then introduce the *homological* Tate spectral sequence pioneered by Bruner-Rognes [21]. This spectral sequence arises from applying homology to the filtration $TP(y(n)) = \lim_i TP(y(n))[i]$, and thus it converges to the continuous homology $H_*^c(TP(y(n)))$ of $TP(y(n))$, by which we mean the inverse-limit of the homology groups $H_*(TP(y(n))[i])$. This continuous homology serves as input to the "localized inverse-limit Adams spectral sequence" converging to the continuous Morava K-theory

$$K(m)_*^c(TP(y(n))) := \lim K(m)_*(TP(y(n))[i]).$$

We discuss this spectral sequence and related spectral sequences, prove that the E_2-term vanishes for $1 \leq m \leq n$.

In Section 5, we carry out a similar analysis for topological negative cyclic homology of $y(n)$, defined as the $\mathbb{T}$-homotopy fixed points

$$TC^{-}(y(n)) := THH(y(n))^{h\,\mathbb{T}}.$$

Using the same homological approach, we compute the continuous Morava K-theory of topological negative cyclic homology of $y(n)$.

In Section 6, we use the equalizer formula for topological cyclic homology

$$TC(y(n)) \longrightarrow TC^{-}(y(n)) \underset{\psi_p}{\overset{\text{can}}{\rightrightarrows}} \widehat{TP}(y(n))$$

due to Nikolaus-Scholze [43] to compute the Morava K-theory $K(m)_*(TC(y(n), H\mathbb{F}_2))$. To do so, we apply Conjecture 6.1 (which will be considered in [5]), which implies that

$$K(m)_*^c(F(y(n), H\mathbb{F}_2) \cong K(m)_*(F(y(n), H\mathbb{F}_2))$$

for $F = TP$ or $F = TC^-$ and $0 \leq m \leq n$. We note that only the results from Section 6 rely on Conjecture 6.1.

Finally, we use the Dundas-Goodwillie-McCarthy pullback square [22] to compute the Morava K-theory of the relative algebraic K-theory $K(y(n), H\mathbb{F}_2)$. The computation follows from our computation of the Morava K-theory of $TC(y(n), H\mathbb{F}_2)$, Quillen's identification of $K(H\mathbb{F}_2)$ in [44], Hesselholt-Madsen's identification of $TC(H\mathbb{F}_2)$ in [29], and work of Waldhausen [51, Prop. 2.2].

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2. Families of Thom spectra

In this section, we recall the construction of and essential facts about the Thom spectra $y(n)$ which interpolate between the sphere spectrum and the mod two Eilenberg-MacLane spectrum. We also introduce a family of spectra $z(n)$ which interpolate between the sphere spectrum and the integral Eilenberg-MacLane spectrum. The construction and basic properties of both familes, such as their homology and multiplicative structure, are discussed in Section 2.1. In Section 2.2, we calculate the chromatic complexity of the spectra $y(n)$, $z(n)$, and $z(n)/v_n$; the results are listed in Proposition 2.20.

2.1. **Construction of the Thom spectra $y(n)$ and $z(n)$.** In this section, we recall the construction and some basic facts about the spectra $y(n)$ and introduce the spectra $z(n)$. We begin by recalling Mahowald's construction of $H\mathbb{F}_2$ and $H\mathbb{Z}$ as Thom spectra [37]. Let $f = \Omega^2 w\colon \Omega^2 S^3 \to \Omega^2 B^3 O \simeq BO$ be the two-fold looping of the generator $w\colon S^3 \to B^3 O$ of $\pi_3(B^3 O) \cong \pi_0(O) \cong \mathbb{Z}/2$. Recall that for an E_∞-ring spectrum R, one can construct the group-like E_∞-space $GL_1 R$ [40, 3]. When $R = S$ is the sphere spectrum, its delooping $BGL_1 S$ is a model for the classifying space of stable spherical fibrations. The classical J homorphism then gives a map of group-like E_∞-spaces $J\colon O \to GL_1 S$. In [37, Sec. 2.6], Mahowald showed that

$$H\mathbb{F}_2 \simeq Th(\Omega^2 S^3 \xrightarrow{f} BO \xrightarrow{BJ} BGL_1 S) \tag{1}$$

where $Th(-)$ is the Thom spectrum construction. (See [2] for a modern reference on this construction that has all the desired properties.) Similarly, Mahowald [37, Prop. 2.8] proved that

$$H\mathbb{Z} \simeq Th(\Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^3 \xrightarrow{f} BO \xrightarrow{BJ} BGL_1 S)$$

where $S^3\langle 3\rangle$ is the fiber of the map $S^3 \to K(\mathbb{Z}, 3)$ and $\iota\colon S^3\langle 3\rangle \to S^3$ is the inclusion of the fiber.

We now produce a family of spectra that interpolate between the sphere spectrum and $H\mathbb{F}_2$ following [38, Def. 3.2]. Recall that the James splitting produces an equivalence $\Omega\Sigma S^2 \simeq J_\infty S^2$ where $J_\infty X$ is the James construction of the space X. Therefore we can rewrite (1) as $H\mathbb{F}_2 \simeq Th(\Omega J_\infty S^2 \to BGL_1 S)$. By truncating the James construction, one can define spectra $J_k S^2 = \bigvee_{i=0}^{k} (S^2)^{\times i}/\sim$, and there is an obvious inclusion $i_k\colon J_k S^2 \hookrightarrow J_\infty S^2$. Taking $k = 2^n - 1$, composition with the inclusion allows us to define

$$y(n) := Th(\Omega J_{2^n-1} S^2 \xrightarrow{f_n} BGL_1 S)$$

where $f_n = BJ \circ f \circ i_{2^n-1}$. Note that one needs to p-localize in order to construct $y(n)$ at odd primes, but this is not necessary at the prime 2.

There is a fiber sequence $J_{2^n-1}S^2 \to \Omega S^3 \to \Omega S^{2^n+1}$ and consequently the map $J_{2^n-1}S^2 \to \Omega S^3$ is $(2^{n+1}-2)$-connected. Thus there is a map $J_{2^n-1}S^2 \to K(\mathbb{Z},2)$ given by truncating homotopy groups and this map is compatible with the map $J_\infty S^2 \to K(\mathbb{Z}, 2)$.

Construction 2.1. Let $n \geq 1$. Write $J_{2^n-1}S^2\langle 2\rangle$ for the fiber of the map $J_{2^n-1}S^2 \to K(\mathbb{Z}, 2)$ given by truncating homotopy groups. Then define

$$z(n) := Th(\Omega(J_{2^n-1}S^2\langle 2\rangle) \xrightarrow{g_n} BGL_1 S)$$

where $g_n = f_n \circ \iota_{2^n-1}$ where $\iota_k \colon J_kS^2\langle 2\rangle) \to J_kS^2$ is the inclusion of the fiber. Note that there is a commutative diagram

$$\begin{tikzcd}
\Omega(J_{2^n-1}S^2\langle 2\rangle) \arrow[r] \arrow[d, "\iota_{2^n-1}"'] & \Omega^2S^3\langle 3\rangle \arrow[d, "\iota"'] \arrow[dr, "g_n"] & \\
\Omega(J_{2^n-1}S^2) \arrow[r, "i_{2^n-1}"] \arrow[d] & \Omega^2S^3 \arrow[r, "f_n"] \arrow[d] & BGL_1S \\
S^1 \arrow[r, "="] & S^1 &
\end{tikzcd}$$

where the left two columns are fiber sequences, all the maps in the upper right triangle are 2-fold loop maps and all the maps in the upper left square are 1-fold loop maps.

Lemma 2.2. *The spectra $z(n)$ are E_1-ring spectra and the diagram*

$$\begin{tikzcd}
z(n) \arrow[r] \arrow[d] & H\mathbb{Z} \arrow[d] \\
y(n) \arrow[r] & H\mathbb{F}_2
\end{tikzcd}$$

is a commutative diagram in the category of E_1-ring spectra.

Proof. This immediately follows from Lewis's theorem [24, Ch. IX] and Construction 2.1. □

The homology of the spectra $y(n)$ interpolate between the homology of the sphere spectrum $S \simeq y(0)$ and the homology of the mod 2 Eilenberg-MacLane spectrum $H\mathbb{F}_2 \simeq y(\infty)$. Similarly, the homology of the spectra $z(n)$ interpolates between the homology of $z(1)$ and the homology of the integral Eilenberg-MacLane spectrum $z(\infty) = H\mathbb{Z}$. More generally, we have the following ladder of interpolations between the sphere spectrum and the mod 2 and integral Eilenberg-MacLane spectra:

$$\begin{tikzcd}
z(1) \arrow[r] \arrow[d] & z(2) \arrow[r] \arrow[d] & \cdots \arrow[r] & z(\infty) = H\mathbb{Z} \arrow[d] \\
y(1) \arrow[r] & y(2) \arrow[r] & \cdots \arrow[r] & y(\infty) = H\mathbb{F}_2.
\end{tikzcd}$$

Notation 2.3. Let $\bar{\xi} := \chi(\xi)$ where χ is the antipode map in the dual Steenrod algebra $\mathcal{A}_*$.

Lemma 2.4. *There are isomorphisms*

$$H_*(y(n)) \cong P(\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n),$$
$$H_*(z(n)) \cong P(\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n)$$

of sub-$\mathcal{A}_*$-comodule algebras of $H_*(H\mathbb{F}_2)$ and $H_*(H\mathbb{Z})$, respectively, for $n \geq 1$. The map $z(n) \to y(n)$ induces the evident inclusion in homology.

Proof. By the Thom isomorphism, $H_*(y(n)) \cong H_*(\Omega J_{2^n-1}S^2)$ and $H_*(z(n)) \cong H_*(\Omega(J_{2^n-1}S^2\langle 2\rangle))$. The Serre spectral sequence arising from the path-loops fibration has the form

$$E_2 = H_*(J_{2^n-1}S^2; H_*(\Omega J_{2^n-1}S^2)) \Rightarrow H_*(PJ_{2^n-1}S^2) = \mathbb{F}_2\{1\}.$$

The homology $H_*(J_{2^n-1}S^2; \mathbb{F}_2)$ can be computed by induction on k from the long exact sequence associated to the cofiber sequence

$$J_{k-1}S^2 \to J_kS^2 \to J_kS^2/J_{k-1}S^2 \simeq (S^2)^{\wedge k}$$

as in [31, Lect. 3]. We have an additive isomorphism

$$H_*(J_{2^n-1}S^2; \mathbb{F}_2) \cong P_{2^n}(x)$$

where $|x| = 2$. In order for the relevant classes to die in the spectral sequence, we must have

$$H_*(\Omega J_{2^n-1}S^2) \cong P(\bar{\xi}_1, \ldots, \bar{\xi}_n).$$

Thus, by the Thom isomorphism, $H_*(y(n)) \cong P(\bar{\xi}_1, \ldots, \bar{\xi}_n)$. Now we compute the analogous Serre spectral sequence

$$E_2 = H_*(J_{2^n-1}S^2\langle 2\rangle; H_*(\Omega(J_{2^n-1}S^2\langle 2\rangle))) \Rightarrow H_*(PJ_{2^n-1}S^2) = \mathbb{F}_2\{1\}.$$

where the only difference is that $H_*(J_{2^n-1}S^2\langle 2\rangle) \cong \mathbb{F}_2\{1, x^2, \ldots, x^{2^{n+1}-1}\}$ so the computation is the same except that $\bar{\xi}_1$ does not arise. Thus, we have $H_*(z(n)) \cong P(\bar{\xi}_1^2, \bar{\xi}_2, \ldots)$. To prove the claimed $\mathcal{A}_*$-coaction and algebra structure, notice that the maps $H_*(y(n)) \to H_*(H\mathbb{F}_p)$ and $H_*(z(n)) \to H_*(H\mathbb{Z})$ are monomorphisms of $\mathcal{A}_*$-comodule algebras. This may be deduced by the map of Serre spectral sequences induced by the map of fiber sequences in Construction 2.1. □

We will often use the unit map $S \to y(n)$ in our computations below, as well as the map $y(n) \to H\mathbb{F}_2$ induced by the inclusion $J_{2^n-1}S^2 \hookrightarrow J_\infty S^2 \simeq \Omega S^3$. Throughout, whenever we write a map $y(n) \to H\mathbb{F}_2$ without decoration we are referring to this map. We record a useful property of this map in the following lemma.

Lemma 2.5. *The map $y(n) \to H\mathbb{F}_2$ is $(2^{n+1} - 2)$-connected.*

Proof. The Adams spectral sequence converging to $y(n)_*$ has the form

$$\mathrm{Ext}^{**}_{\mathcal{A}_*}(\mathbb{F}_2, P(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n)) \Rightarrow y(n)_*$$

and the Adams spectral sequence converging to $(H\mathbb{F}_2)_*$ has the form

$$\mathrm{Ext}^{**}_{\mathcal{A}_*}(\mathbb{F}_2, P(\bar{\xi}_1, \bar{\xi}_2, \ldots)) \cong \mathbb{F}_2 \Rightarrow (H\mathbb{F}_2)_* = \mathbb{F}_2.$$

Let $C_\bullet(n)$ denote the cobar complex whose homology is the first Ext-group, and let $C_\bullet(\infty)$ denote the cobar complex whose homology is the second Ext-group. These complexes differ whenever $\bar{\xi}_r$ for $r \geq n+1$ appears in $C_\bullet(\infty)$. Since $|\bar{\xi}_r| \geq |\bar{\xi}_{n+1}| = 2^{n+1} - 1$, the resulting homologies agree up to degree $2^{n+1} - 2$. Since the second spectral sequence collapses at E_2, we conclude that $\pi_i(y(n)) = 0$ for $i \leq 2^{n+1} - 2$, which proves the lemma. □

Using the fact that $z(n)$ is an E_1-ring spectrum, we construct a v_n-self map of $z(n)$ that will be important later.

Lemma 2.6. *The spectrum $z(n)$ has a v_n-self map.*

Proof. We compute the Adams spectral sequence for $z(n)$ in a range. First we describe the input of the Adams spectral sequence. Following Mahowald-Ravenel-Schick, we will write $B(n)_*$ for the Hopf algebra in the extension

$$H_*(y(n)) \to \mathcal{A}_* \to B(n)_*$$

and we will call $C(n)_*$ the Hopf algebra in the extension

$$H_*(z(n)) \to \mathcal{A}_* \to C(n)_*.$$

We also have a Hopf algebra extension

$$B(n)_* \to C(n)_* \to E(\bar{\xi}_1)$$

and an associated Cartan-Eilenberg spectral sequence

$$\mathrm{Ext}^{*,*}_{B(n)}(\mathbb{F}_2, \mathrm{Ext}_{E(\bar{\xi}_1)}(\mathbb{F}_2, \mathbb{F}_2)) \cong \mathrm{Ext}^{*,*}_{B(n)}(\mathbb{F}_2, P(h_0)) \Rightarrow \mathrm{Ext}^{*,*}_{C(n)_*}(\mathbb{F}_2, \mathbb{F}_2)$$

Since h_0 is in degree 1 and since there are no classes in $\mathrm{Ext}^k_{B(n)}(\mathbb{F}_2, \mathbb{F}_2)$ in adjacent degrees for $k \leq 2^{n+1}$, the Cartan-Eilenberg spectral sequence collapses in this range. For degree reasons the Adams spectral sequence also collapses in this range and we have a class v_n in $\pi_{2^{n+1}-2}(z(n))$ which supports an h_0-tower in the spectral sequence.

Since there exists an element $v_n \in \pi_{2^{n+1}-2}(z(n))$ and $z(n)$ is an E_1-ring spectrum we can produce a v_n-self map as the composite

$$\Sigma^{2^{n+1}-2} S \wedge z(n) \to z(n) \wedge z(n) \to z(n)$$

where the first map in the composite is $v_n \wedge \mathrm{id}_{z(n)}$ and the second map is the multiplication map of $z(n)$. □

Remark 2.7. In fact, we will later use Margolis homology to show that v_n generates a copy of $\mathbb{Z}_2$ in the abutment of the Adams spectral sequence for $z(n)$.

Lemma 2.8. *Let $z(n)/v_n$ denote the cofiber of v_n. Then we have an isomorphism of $\mathcal{A}_*$-comodules*

$$H_*(z(n)/v_n) \cong H_*(z(n)) \otimes E(\bar{\xi}_{n+1}).$$

Proof. The cofiber sequence

$$\Sigma^{2^{n+1}-2} z(n) \xrightarrow{v_n} z(n) \longrightarrow z(n)/v_n$$

gives rise to a long exact sequence in cohomology

$$\cdots \to H^*(\Sigma^{2^{n+1}-1} z(n)) \to H_*(z(n)/v_n) \to H^*(z(n)) \to H^*(\Sigma^{2^{n+1}-2} z(n)) \to \cdots .$$

By [32, Lem. 3.2] for example, the induced map $H^*(v_n)$ is the zero map and therefore we have an extension

$$0 \to \Sigma^{2^{n+1}-1} H^*(z(n)) \to H^*(z(n)/v_n) \to H^*(z(n)) \to 0.$$

Thus $H^*(z(n)/v_n)$ is an exterior extension of $H^*(z(n))$, i.e. $H^*(z(n)/v_n) \cong H^*(z(n)) \otimes E(x)$ where $|x| = 2^{n+1} - 1$. Since $K(n)_*(v_n)$ is an isomorphism, the map

$$k(n)[0, 2^{n+1} - 2]^0(z(n)) \to k(n)[0, 2^{n+1} - 2]^0(\Sigma^{2^{n+1}-2} z(n))$$

is also an isomorphism and therefore the map $z(n)/v_n \to k(n)[0, 2^{n+1} - 2]$ is nullhomotopic. We then examine the commutative diagram

$$\begin{tikzcd}
z(n) \arrow[r] \arrow[d] & z(n)/v_n \arrow[d] \arrow[dr] & \\
k(n)[0, 2^{n+1} - 2] \arrow[r] & H\mathbb{F}_2 \arrow[r, "Q_n"] & \Sigma^{2^{n+1}-2} H\mathbb{F}_2
\end{tikzcd}$$

which we have since the first k-invariant of $k(n)$ is Q_n. Since the map $z(n)/v_n \to k(n)[0, 2^{n+1} - 2]$ is null-homotopic the map $z(n)/v_n \to \Sigma^{2^{n+1}-2} H\mathbb{F}_2$ is not null-homotopic and represents a nontrivial class $Q_n(1) \in H^{2^{n+1}-2}(z(n)/v_n)$. Since this is also how the class x arises, the class $x = Q_n$. Thus, $H_*(z(n)/v_n) \cong \mathrm{Hom}_{\mathbb{F}_2}(H^*(z(n)/v_n), \mathbb{F}_2) \cong H_*(z(n)) \otimes E(\bar{\xi}_{n+1})$. □

2.2. **Chromatic complexity of $y(n)$, $z(n)$, and $z(n)/v_n$.** In this section, we calculate the chromatic complexity of $y(n)$, $z(n)$, and $z(n)/v_n$. We begin with a recollection of Morava K-theory and discuss its calculation via the localized Adams spectral sequence and Margolis homology. We then calculate the Margolis homology groups of the above Thom spectra to compute their Morava K-theory; the results are listed in Proposition 2.20.

Connective Morava K-theory $k(m)$ has homotopy groups $k(m)_* \cong \mathbb{F}_2[v_m]$. Inverting v_m defines *Morava K-theory* $K(m)$ with homotopy groups $K(m)_* \cong \mathbb{F}_p[v_m^{\pm 1}]$.

Definition 2.9. Let X be a p-local spectrum. Then X has *type* m if $m \geq 0$ is the smallest nonnegative integer for which $K(m)_*(X)$ is nontrivial.

Remark 2.10. Note that we are applying the definition of type to spectra which are not finite. This means that there are some spectra which should be thought of as chromatic complexity n, which are not type n. This definition is therefore useful only in specific situations. For example, p-local complex K-theory $KU_{(p)}$ is usually considered to have chromatic complexity 1 because its associated formal group law is height 1, but it is rationally nontrivial and therefore does not have type 1.

Although the spectra $y(n)$ are not finite complexes, they are still type n by work of [38] and therefore using type as our notion of chromatic height is useful for these spectra.

Recall that $H_*(k(m)) \cong \mathcal{A}_* \square_{E(\bar{\xi}_{m+1})} \mathbb{F}_2$ and $\bar{\xi}_{m+1}$ is dual to the m-th Milnor primitive Q_m. The Adams spectral sequence converging to $k(m)_*(X)$ has the form

$$E_2 = \operatorname{Ext}^{**}_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*(X)) \Rightarrow k(m)_*(X)$$

by the Künneth isomorphism and the change-of-rings isomorphism.

Morava K-theory $K(m)$ can be obtained from $k(m)$ by inverting v_m, i.e. $K(m)$ can be constructed as the telescope

$$K(m) = \widehat{k(m)} = \varinjlim \left(k(m) \xrightarrow{v_m} \Sigma^{-2p^m+2} k(m) \xrightarrow{v_m} \cdots \right).$$

The homotopy groups of a telescope can be computed using the localized Adams spectral sequence [42]. We will discuss the convergence of this spectral sequence below; for now we only consider its E_2-page. The E_2-page of this spectral sequence can be computed by inverting v_m at the level of Ext-groups as in [38, Section 2.5] and [23]. For a p-complete spectrum X, one obtains a spectral sequence

$$v_m^{-1} E_2 = v_m^{-1} \operatorname{Ext}^{**}_{E(Q_m)}(H^*(X), \mathbb{F}_2) \Rightarrow \widehat{k(m)}_*(X) \cong K(m)_*(X).$$

Note that we have only used the v_m-map on $k(m)$, so there is no decoration on X.

In order to compute $v_m^{-1} E_2$, we will use Margolis homology [39, Ch. 19] which encodes the action of the Milnor primitive Q_m on $H^*(X)$.

Definition 2.11. Let M be a module over $E(Q_m)$. Since Q_m is an exterior generator, we obtain a complex

$$M \xrightarrow{Q_m} M \xrightarrow{Q_m} M.$$

The homology of this complex $H(M; Q_m)$ is the *Margolis homology of M with respect to Q_m*. If X is a spectrum, then the *Margolis homology of X with respect to Q_m*, denoted $H(X; Q_m)$, is defined by taking $M = H_*(X)$.

The following lemma says that we could equivalently define $H(X; Q_m)$ by taking $M = H^*(X)$ in Definition 2.11 and then dualizing.

Lemma 2.12. *Let X be a spectrum with $H^*(X)$ finite type. There is a natural isomorphism of (left) $E(Q_m)$-modules*

$$H(H^*(X), Q_m) \cong D(H(X, Q_m))$$

where $D(-)$ is the dual $\mathcal{A}$-module.

Proof. For P^t_s-Margolis homology, the result follows from [39, Ch. 19, Prop. 12] by letting $M = H_*(X)$ since in this case there is an isomorphism $D(M) \cong H^*(X)$. For Q_m-homology, the proof is the same. $\square$

Lemma 2.13. *Let $m \geq 1$. If $H(X; Q_m) = 0$, then $v_m^{-1} E_2 = 0$.*

Proof. We use a specialization of [39, Ch. 18, Cor. 16] to the case where the base exterior algebra is generated by one generator Q_m. Using Margolis' notation, we have $E(e_1) := E(Q_m)$. Therefore, when $p = 1$, we see that $H(X; e_1) = 0$ and the second condition is vacuous. Thus, $\operatorname{Ext}^j_{E(Q_m)}(H^*(X), \mathbb{F}_2) = 0$ for $j > 0$. In particular, the spectral sequence collapses to the 0-line. Since multiplication by v_m increases Adams filtration by at least one, we conclude that

$$v_m^{-1} E_2 = v_m^{-1} \operatorname{Ext}_{E(Q_m)}(H^*(X), \mathbb{F}_2) = 0.$$

$\square$

The action of Q_m on the generator $\bar{\xi}_k \in \mathcal{A}_*$ can easily be computed using the coproduct $\psi\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$. In particular,

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}$$

so we have

$$Q_m(\bar{\xi}_k) = \begin{cases} \bar{\xi}_{k-m-1}^{2^{m+1}} & k \geq m+1, \\ 0 & else \end{cases} \tag{2}$$

where by convention $\bar{\xi}_0 = 1$. This action can be extended to all of $\mathcal{A}_*$ using the fact that Q_m acts as a derivation. We now compute the Margolis homology of the dual Steenrod algebra. The chain complexes defined in the proof will be used in our computation of $H(y(n); Q_m)$ below.

Lemma 2.14. *The Margolis homology of the dual Steenrod algebra $H(\mathcal{A}_*; Q_m)$, or equivalently the Margolis homology of $H\mathbb{F}_2$, vanishes for all $m \geq 0$.*

Proof. This is [39, Ch. 19, Prop. 1], but our proof is modeled after [1, Lem. 16.9]. We begin with $H(\mathcal{A}_*; Q_0)$, which is somewhat exceptional. We express $\mathcal{A}_*$ as the tensor product of the chain complexes (with differential Q_0)

$$\begin{aligned} &(e_0) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_1\}, \\ &(c_r) \quad \mathbb{F}_2\{1, \bar{\xi}_r^2\} \leftarrow \mathbb{F}_2\{\bar{\xi}_{r+1}, \bar{\xi}_r^4\} \leftarrow \mathbb{F}_2\{\bar{\xi}_r^2\bar{\xi}_{r+1}, \bar{\xi}_r^6\} \leftarrow \mathbb{F}_2\{\bar{\xi}_r^4\bar{\xi}_{r+1}, \bar{\xi}_r^8\} \leftarrow \cdots, \end{aligned}$$

where $r \geq 1$. Each chain complex (c_r) has homology $\mathbb{F}_2\{1\}$ and the chain complex (e_0) has vanishing homology, so by the Künneth isomorphism for Margolis homology [39, Ch. 19, Prop. 18], we have $H(\mathcal{A}_*; Q_0) \cong 0$.

Now we compute $H(\mathcal{A}_*; Q_1)$. We can decompose $\mathcal{A}_*$ as the tensor product of the chain complexes (with differential Q_1)

$$\begin{aligned} &(e_0) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_2\}, \\ &(c_r) \quad \mathbb{F}_2\{1, \bar{\xi}_r^4\} \leftarrow \mathbb{F}_2\{\bar{\xi}_{r+2}, \bar{\xi}_r^8\} \leftarrow \mathbb{F}_2\{\bar{\xi}_r^4\bar{\xi}_{r+2}, \bar{\xi}_r^{12}\} \leftarrow \mathbb{F}_2\{\bar{\xi}_r^8\bar{\xi}_{r+2}, \bar{\xi}_r^{16}\} \leftarrow \cdots, \\ &(d_1) \quad \mathbb{F}_2\{1, \bar{\xi}_1, \bar{\xi}_1^2\}, \\ &(d_s) \quad \mathbb{F}_2\{1, \bar{\xi}_s^2\}, \end{aligned}$$

where $r \geq 1$ and $s \geq 2$. The chain complex (e_0) has vanishing homology, so by the Künneth isomorphism we have $H(\mathcal{A}_*; Q_1) \cong 0$.

The computation of $H(\mathcal{A}_*; Q_m)$ for $m \geq 2$ is similar. One decomposes $\mathcal{A}_*$ into chain complexes as above; the chain complex

$$(e_0) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_{m+1}\}$$

will have vanishing homology, so $H(\mathcal{A}_*; Q_m) = 0$. □

Corollary 2.15. *The Margolis homology of $(A//A(0))^\vee \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2, \ldots]$, or equivalently the Margolis homology of $H\mathbb{Z}$, is given by*

$$H(H\mathbb{Z}; Q_m) \cong \begin{cases} \mathbb{F}_2 & m = 0, \\ 0 & else. \end{cases}$$

Proof. We begin with $m = 0$. The only difference between this computation and the computation for $H_*(H\mathbb{F}_2, Q_0)$, is that we remove the chain complex (e_0). Since the homology of the remaining complexes (c_r) is $\mathbb{F}_2$ in each case, $H_*(H\mathbb{Z}, Q_0) \cong \mathbb{F}_2$.

For $m > 0$, we can use the same complexes as in the previous proof after replacing (d_1) by the chain complex $\mathbb{F}_2\{1, \bar{\xi}_1^2\}$. □

We can compute the Margolis homology of $y(n)$ and $z(n)$ by modifying these complexes further.

Lemma 2.16. *The Margolis homology of $P(\bar{\xi}_1, \dots, \bar{\xi}_n)$, or equivalently the Margolis homology of $y(n)$, is given by*

$$H(y(n); Q_m) \cong \begin{cases} 0 & \text{if } 0 \leq m \leq n-1, \\ H_*(y(n)) & \text{if } m \geq n. \end{cases}$$

Proof. When $m = 0$, the first r for which the complex (c_r) cannot be defined is (c_n) since $\bar{\xi}_{n+1} \notin H_*(y(n))$. Therefore we replace (c_n) by the complex

$$(c'_n) \quad \mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_n^2\} \leftarrow \mathbb{F}_2\{\bar{\xi}_n^4\} \leftarrow \mathbb{F}_2\{\bar{\xi}_n^6\} \leftarrow \cdots .$$

Then $H_*(y(n))$ decomposes as the tensor product of the complexes (e_0), $(c_r)_{1 \leq r \leq n-1}$, and (c'_n). The homology of (c'_n) is nontrivial but the homology of (e_0) vanishes, so by the Künneth isomorphism we conclude that $H(y(n); Q_0) \cong 0$.

When $1 \leq m \leq n-1$, we make a similar change. We end up with redefined chain complexes (c'_r) for $n - m \leq r \leq n$. Since we still tensor with the acyclic complex (e_0), we still have $H(y(n); Q_m) = 0$.

When $m \geq n$, we no longer include the chain complex (e_0) since $\bar{\xi}_{n+1} \notin H_*(y(n))$. Since $Q_n(\bar{\xi}_i) = 0$ for all $1 \leq i \leq n$, we see that $H_*(y(n))$ is generated by cycles and obtain the desired isomorphism. □

The same proof applied to the complexes used to compute $H(H\mathbb{Z}; Q_m)$ gives a $z(n)$-analog of the previous result.

Corollary 2.17. *The Margolis homology of $P(\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n)$, or equivalently the Margolis homology of $z(n)$, is given by*

$$H(z(n); Q_m) \cong \begin{cases} P(\bar{\xi}_n^2) & \text{if } m = 0, \\ 0 & \text{if } 1 \leq m \leq n-1, \\ H_*(z(n)) & \text{if } m \geq n. \end{cases}$$

Proof. We do the same alterations to Lemma 2.16 as we did to produce the proof of Corollary 2.15 from Lemma 2.14, so we will just describe the case $m = 0$. We use the same chain complexes as in Lemma 2.16 except we do not include the acyclic complex (e_0). Thus, the Margolis homology is a tensor product of copies of $\mathbb{F}_2$ with the homology of (c_n). Thus, $H_*(z(n); Q_0) \cong P(\bar{\xi}_n^2)$. □

Remark 2.18. Note that Corollary 2.17 implies that the elements $v_n^k \in \pi_*(z(n))$ generate copies of the integers in $z(n)$. We claim that after coning off v_n all of the copies of the integers in $\pi_*(z(n))$ are killed except for the copy of the integers in degree zero; this follows from the next result. In other words, the map $z(n)/v_n \to H\mathbb{Z}$ is a rational equivalence. We will return to this point later.

Corollary 2.19. *The Margolis homology of $P(\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n) \otimes E(\bar{\xi}_{n+1})$, or equivalently the Margolis homology of $z(n)/v_n$, is given by*

$$H(z(n)/v_n; Q_m) \cong \begin{cases} \mathbb{F}_2 & \text{if } m = 0, \\ 0 & \text{if } 1 \leq m \leq n, \\ H_*(z(n)/v_n) & \text{if } m \geq n+1. \end{cases}$$

Proof. The proof in the case $m = n$ follows by tensoring the complexes from the previous corollary with the complex

$$\mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_{n+1}\}.$$

Since this complex is Q_n-acyclic, we observe that $H(z(n)/v_n; Q_n) \cong 0$. For $m = 0$, we make the following adjustment. Rather than replacing (c_n) with (c'_n) as in Lemma 2.16, we keep the complex (c_n) and remove (c_r) for $r > n$. This has the consequence that $H_*(z(n)/v_n, Q_0) \cong \mathbb{F}_2$. In the case

$0 < m < n$, we only replace (c_r) with (c'_r) for $n-m < r \leq n$. The Margolis homology is still trivial because we are tensoring with the acyclic complex $\mathbb{F}_2\{1\} \leftarrow \mathbb{F}_2\{\bar{\xi}_m\}$. The case $m > n$ is exactly the same as in Lemma 2.16 □

We can assemble these Margolis homology computations to compute the E_2-pages of the localized Adams spectral sequence converging to $K(m)_*(X)$ for $X = y(n), z(n)$. We will discuss this spectral sequence in more detail in the following section.

Proposition 2.20. *The chromatic complexity of $y(n)$, $z(n)$ and $z(n)/v_n$ may be described as follows:*

- *The spectrum $y(n)$ is $K(m)$-acyclic for $0 \leq m \leq n-1$, and $K(n)_*(y(n)) \neq 0$.*
- *The spectrum $z(n)$ is $K(m)$-acyclic for $1 \leq m \leq n-1$, and $K(m)_*(z(n)) \neq 0$ for $m = 0, n$.*
- *The spectrum $z(n)/v_n$ is $K(m)$-acyclic for $1 \leq m \leq n$, and $K(m)_*(z(n)/v_n) \neq 0$ for $m = 0$.*

Proof. We give the proof for $y(n)$; the proofs for $z(n)$ and $z(n)/v_n$ are similar. Whenever the Margolis homology $H(y(n); Q_m)$ vanishes, we have

$$v_m^{-1} E_2 = v_m^{-1} \operatorname{Ext}_{E(Q_m)}(H^*(y(n)), \mathbb{F}_2) \cong 0$$

by Lemma 2.13. Therefore Lemma 2.16 proves the $K(m)$-acyclicity of $y(n)$ for $0 \leq m \leq n-1$. When $m = n$, we have

$$v_n^{-1} E_2 = v_n^{-1} \operatorname{Ext}_{E(Q_n)}(H^*(y(n)), \mathbb{F}_2) \cong P(v_n^{\pm 1}) \otimes H(y(n); Q_n).$$

To see that $K(n)_*(y(n)) \neq 0$, it suffices to produce a class which survives the localized Adams spectral sequence. This follows from the collapsing range for the Adams spectral sequence converging to $\pi_*(y(n))$ in [38, Lem. 3.5] together with the Leibniz rule. In the proofs for $z(n)$ and $z(n)/v_n$, one can produce a similar collapsing range from the Hopf algebra extensions in the proof of Lemma 2.6. □

We conclude this section with some material which will be useful in Sections 4 and 5.

Definition 2.21. Let $E(n) := E(Q_0, \ldots, Q_n)$ denote the subalgebra of the Steenrod algebra generated by the first $n+1$ Milnor primitives. We also allow the case $n = \infty$, in which case we simply write

$$\mathcal{E} := E(\infty) = E(Q_0, Q_1, \ldots)$$

for the subalgebra generated by all of the Milnor primitives. As usual, we write $E(n)_*$ and $\mathcal{E}_*$ for the $\mathbb{F}_p$-linear duals of these subalgebras of the Steenrod algebra.

Lemma 2.22. *Suppose that M and N are isomorphic as $E(n)$-modules for $0 \leq n \leq \infty$. Then there is an isomorphism $H(M; Q_i) \cong H(N; Q_i)$ for all $i \leq n$.*

Proof. This is clear from the definition of $H(-; Q_i)$. □

Remark 2.23. When X is a spectrum, we defined $H(X; Q_i)$ to be $H(H_*(X); Q_i)$. Therefore if $H_*(X)$ and $H_*(Y)$ are isomorphic as $E(n)_*$-comodules, then there is an isomorphism $H(M; Q_i) \cong H(N; Q_i)$ for all $i \leq n$.

3. Homology of topological Hochschild homology of $y(n)$

We now turn to the study of the topological Hochschild homology of $y(n)$. We begin by determining the homology of the topological Hochschild homology of $y(n)$ using the Bökstedt spectral sequence [19]. We then analyze the map from the homology of topological Hochschild homology of $y(n)$ to the homology of topological Hochschild homology of $H\mathbb{F}_p$.

Remark 3.1. The calculations in this section and the sequel are complicated by two facts:

(1) We do not know if $THH(y(n))$ admits a ring structure since $y(n)$ is only known to be an E_1-ring spectrum. Therefore we will only prove *additive isomorphisms* throughout the remaining sections since there is no multiplicative structure *a priori* on $H_*(THH(y(n)))$. We do know that $H_*(THH(y(n)))$ is a module over $E(\sigma)$ in the category of $\mathcal{A}_*$-comodules, however, and this structure will be useful for determining the map $H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_p))$.
(2) There is indeterminacy in the names of many classes. For example, classes in $H_*(THH(y(n)))$ are defined only up to lower Bökstedt filtration. This does not change the additive presentation of our calculations, but it does affect deeper structure such as the coaction of the dual Steenrod algebra $\mathcal{A}_*$.

Proposition 3.2. *There is an isomorphism of graded $\mathcal{A}_*$-comodules*

$$H_*(THH(y(n))) \cong H_*(y(n)) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots \sigma\bar{\xi}_n)$$

where the coaction

$$\nu_n \colon H_*(THH(y(n))) \to H_*(THH(y(n))) \otimes \mathcal{A}_*$$

on elements $x \in H_*(y(n))$ is determined by the restriction of the coproduct on $\mathcal{A}_*$ to $H_*(y(n)) \subset \mathcal{A}_*$ and the coaction on $\sigma\bar{\xi}_i$ is determined by the formula $\nu_n(\sigma\bar{\xi}_i) = (1 \otimes \sigma)\nu_n(\bar{\xi}_i)$.

Proof. The Bökstedt spectral sequence has E_2-term

$$E_2^{*,*} \cong HH_*(H_*(y(n))) \cong P(\bar{\xi}_1, \dots, \bar{\xi}_n) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n)$$

which maps injectively to the E_2-term of the Bökstedt spectral sequence for $H\mathbb{F}_2$. The latter spectral sequence is multiplicative and all the algebra generators are concentrated in Bökstedt filtration zero and one. Consequently, the Bökstedt spectral sequence for $H\mathbb{F}_2$ collapses and the injective map of spectral sequences implies that the Bökstedt spectral sequence for $y(n)$ also collapses. Note that in the $H\mathbb{F}_2$-Bökstedt spectral sequence there are multiplicative extensions, but these multiplicative extensions do not necessarily occur in the Bökstedt spectral sequence for $y(n)$ because $THH(y(n))$ is not a ring spectrum.

The Bökstedt spectral sequence is a spectral sequence of $\mathcal{A}_*$-comodules and the formula $\nu_n(\sigma x) = (1\otimes\sigma)\nu_n(x)$ holds because the operator σ is induced by a map of spectra $\mathbb{T} \wedge R \to THH(R)$ (see for example [7, Eq. 5.11]) and this implies the $\mathcal{A}_*$-coaction up to elements in lower Bökstedt filtration. □

This produces an additive interpolation between $H_*(THH(S)) \cong H_*(S) \cong \mathbb{F}_2$ and

$$H_*(THH(H\mathbb{F}_2)) \cong P(\bar{\xi}_i | i \geq 1) \otimes E(\sigma\bar{\xi}_i | i \geq 1)$$

as expected. We will use the fact that the Bökstedt spectral sequence computing $H_*(THH(y(n)))$ agrees with the Bökstedt spectral sequence computing $H_*(THH(H\mathbb{F}_2))$ up until degree $2^{n+1} - 2 = |\bar{\xi}_{n+1}| - 1$. We will also frequently use the map

$$\phi_n \colon THH(y(n)) \to THH(H\mathbb{F}_2)$$

induced by the map of Thom spectra $y(n) \to H\mathbb{F}_2$ in the remainder of this section.

Lemma 3.3. *The map $\phi_n \colon THH(y(n)) \to THH(H\mathbb{F}_2)$ is $(2^{n+1} - 2)$-connected.*

Proof. The functor $THH(-)$ preserves connectivity of maps of E_1-ring spectra. □

The rest of this section is dedicated to studying the induced map on homology

$$\phi_{n*} \colon H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2)).$$

The spectrum $THH(y(n))$ has a canonical circle action $\mathbb{T}_+ \wedge THH(y(n)) \to THH(y(n))$ compatible with a structure map $\sigma \colon \mathbb{T} \wedge y(n) \to THH(y(n))$. Though $THH(y(n))$ may not be a ring spectrum, the structure map σ still acts as though it were a derivation on $H_*(THH(y(n)))$ as in [41, Prop. 3.2]. Indeed, the proof of [41, Prop. 3.2] only relies on R being an E_1-ring spectrum. This behavior will be important for our analysis in the next section since the structure map σ determines the d^2-differentials in the homological $\mathbb{T}$-Tate spectral sequence.

Lemma 3.4. *The σ-operator on $H_*(THH(y(n)))$ satisfies*

$$\phi_{n*}(\sigma(x)) = \sigma(\phi_{n*}(x)).$$

Proof. The structure map σ is natural by construction. ☐

Remark 3.5. If $x \in H_*(THH(y(n)))$ satisfies $\sigma(x) = 0$, we will refer to x as a *σ-cycle.* If $x = \sigma(y)$ for some $y \in H_*(THH(y(n)))$, we will refer to x as a *σ-boundary.* We will often describe subsets of $H_*(THH(y(n)))$ and $H_*THH(H\mathbb{F}_p)$ "up to σ-cycles" in the next section. Since the d^2-differentials in the homological Tate spectral sequence and homological homotopy fixed point spectral sequence are given by the action of σ, we will be able to ignore these possible σ-cycles when computing topological periodic cyclic homology and topological negative cyclic homology in the sequel.

Lemma 3.6. *Let $x \in H_*(THH(y(n)))$. The induced map on homology*

$$\phi_{n*}\colon\ H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$$

has the form

$$\phi_{n*}(x) = x + \text{(classes of lower Bökstedt filtration)}.$$

Proof. The linearization map $y(n) \to H\mathbb{F}_2$ induces a map of Bökstedt spectral sequences. The lemma follows from the construction of the Bökstedt spectral sequence. ☐

The $\mathcal{A}_*$-coaction on $H_*(THH(y(n)))$ will be denoted

$$\nu_n\colon\ H_*(THH(y(n))) \to \mathcal{A}_* \otimes H_*(THH(y(n)))$$

for $0 \le n \le \infty$ with the convention that $y(\infty) = H\mathbb{F}_p$.

Lemma 3.7. [7, Eq. 5.11] *Let $x \in H_*(THH(y(n)))$. Then $\nu_n(\sigma(x)) = (1 \otimes \sigma)\nu_n(x)$ for all $0 \le n \le \infty$.*

Proof. This follows because the operator σ is induced by a map of spectra $\mathbb{T} \wedge y(n) \to THH(y(n))$. ☐

Lemma 3.8. *The induced map on homology $\phi_{n*}\colon\ H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$ is a map of $\mathcal{A}_*$-comodules; i.e. the formula $(\mathrm{id} \otimes \phi_{n*}) \circ \nu_n = \nu_\infty \circ \phi_{n*}$ holds.*

Proof. This is true for any map in homology induced by a map of spectra. ☐

We thank Vigleik Angeltveit for discussions which led to a simplification of the proof of Proposition 3.12. We also borrow notation from [6, Prop. 4.12] where he proves a result which bears some resemblance to our own. We also note once and for all that elements in $H_*(THH(y(n)))$ are only well-defined up to lower Bökstedt filtration as in [6, Sec. 5]. Consequently, we may regard $\sigma\bar{\xi}_i$ as a comodule primitive for all i since, up to lower Bökstedt filtration, these elements are comodule primitives.

Before stating Proposition 3.12, we include the following example to illustrate some subtleties in understanding ϕ_{n*}.

Example 3.9. We will fully describe the map

$$P(\bar{\xi}_1) \otimes E(\sigma\bar{\xi}_1) \cong H_*(THH(y(1))) \xrightarrow{\phi_{1*}} H_*(THH(H\mathbb{F}_2)) \cong P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes P(\sigma\bar{\xi}_1)$$

induced by the map $\phi_1\colon\ THH(y(1)) \to THH(H\mathbb{F}_2)$ as a map of $E(\sigma)$-modules in the category of $\mathcal{A}_*$-comodules. We first describe this map as map of $E(\sigma)$-modules.

By Lemma 3.6, we have $\phi_{1*}(\bar{\xi}_1^i) = \bar{\xi}_1^i$ since $\bar{\xi}_1^i \in H_i(THH(y(1)))$ has Bökstedt filtration zero. In general, the map ϕ_{1*} sends classes in $H_*(THH(y(n)))$ in Bökstedt filtration zero to the classes with the same name in $H_*(THH(H\mathbb{F}_2))$.

Moving on to Bökstedt filtration one, we know that either

$$\phi_{1*}(\sigma\bar{\xi}_1) = \sigma\bar{\xi}_1 \text{ or } \phi_{1*}(\sigma\bar{\xi}_1) = \sigma\bar{\xi}_1 + \bar{\xi}_1^2$$

for degree reasons. If the latter formula holds, we may simply change our basis for the vector space $H_2THH(y(1)) \cong \mathbb{F}_p\{\sigma\bar{\xi}_1, \bar{\xi}_1^2\}$ to account for this, so we may assume the former.

We now analyze the key case. Consider the class $\bar{\xi}_1\sigma\bar{\xi}_1 \in H_3(THH(y(1)))$. We claim that $\phi_{1*}(\bar{\xi}_1\sigma\bar{\xi}_1) \neq \bar{\xi}_1\sigma\bar{\xi}_1$. In fact, we know that $\sigma(\bar{\xi}_1\sigma\bar{\xi}_1) = 0$ in $H_*(THH(y(1)))$ and therefore $\bar{\xi}_1\sigma\bar{\xi}_1$ must map to a σ-cycle in $H_*(THH(H\mathbb{F}_2))$. By Lemma 3.6, we know that $\bar{\xi}_1\sigma\bar{\xi}_1$ maps to the class of the same name modulo classes in lower Bökstedt filtration. We also know that in $H_*(THH(H\mathbb{F}_p))$ $\sigma(\bar{\xi}_1\sigma\bar{\xi}_1) = \sigma\bar{\xi}_2$. Therefore, $\phi_{1*}(\bar{\xi}_1\sigma\bar{\xi}_1) = \bar{\xi}_1\sigma\bar{\xi}_1 + y$ where y is in Bökstedt filtration zero and $\sigma y = \sigma\bar{\xi}_2$. The only such element in $H_*(THH(H\mathbb{F}_p))$ with these properties is $\bar{\xi}_2$ itself. Thus,

$$\phi_{1*}(\bar{\xi}_1\sigma\bar{\xi}_1) = \bar{\xi}_1\sigma\bar{\xi}_1 + \bar{\xi}_2.$$

We then claim that $\phi_{1*}(\bar{\xi}_1^{2k}\sigma\bar{\xi}_1) = \bar{\xi}_1^{2k}\sigma\bar{\xi}_1$. We know that $\sigma(\bar{\xi}_1^{2k}\sigma\bar{\xi}_1) = 0$ in both the source and target. Therefore, the only possibility is that we add σ-cycles in either the source or target of the map. Since this does does not affect the map up to isomorphism of $E(\sigma)$-modules, we may assume $\phi_{1*}(\bar{\xi}_1^{2k}\sigma\bar{\xi}_1) = \bar{\xi}_1^{2k}\sigma\bar{\xi}_1$.

We also claim that $\phi_{1*}(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1) = \bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1 + \bar{\xi}_1^{2k}\bar{\xi}_2$. Again, we know $\sigma(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1) = 0$ in $H_*(THH(y(1)))$ whereas $\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1 = \bar{\xi}_1^{2k}\sigma\bar{\xi}_2$ in $H_*(THH(H\mathbb{F}_p))$. Therefore, we must add a term y in Bökstedt filtration zero such that $\sigma y = \bar{\xi}_1^{2k}\sigma\bar{\xi}_2$ and the only possibility is $\bar{\xi}_1^{2k}\sigma\bar{\xi}_2$. This completely determines the map up to isomorphism of $E(\sigma)$-modules.

We now describe the map as a map of $E(\sigma)$-modules in the category of $\mathcal{A}_*$-comodules up to some indeterminacy. First, note that there are no σ-cycles in the degree of $\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1$ in lower Bökstedt filtration and thus we know the answer for $\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1$ completely as a map of $\mathcal{A}_*$-comodules. This also forces the $\mathcal{A}_*$-comodule structure on these elements. For example, since

$$\begin{aligned}
\nu_\infty(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1 + \bar{\xi}_1^{2k}\bar{\xi}_2) \quad &= \quad (\bar{\xi}_1 \otimes 1 + 1 \otimes \bar{\xi}_1)^{2k+1}(1 \otimes \sigma\bar{\xi}_1) + \\
&\quad\quad (\bar{\xi}_1^{2k} \otimes 1 + 1 \otimes \bar{\xi}_1^{2k})(\bar{\xi}_2 \otimes 1 + \bar{\xi}_1 \otimes \bar{\xi}_1^2 + 1 \otimes \bar{\xi}_2) \\
&= \quad (\bar{\xi}_1 \otimes 1 + 1 \otimes \bar{\xi}_1)^{2k+1}(1 \otimes \sigma\bar{\xi}_1) + \bar{\xi}_1^{2k}\bar{\xi}_2 \otimes 1 + \\
&\quad\quad \bar{\xi}_2 \otimes \bar{\xi}_1^{2k} + \bar{\xi}_1^{2k+1} \otimes \bar{\xi}_1^2 + \bar{\xi}_1^{2k+3} \otimes 1 + \bar{\xi}_1^{2k} \otimes \bar{\xi}_2 + 1 \otimes \bar{\xi}_1^{2k}\bar{\xi}_2,
\end{aligned}$$

we know that

$$\nu_1(\bar{\xi}_1^{2k+1}\sigma\bar{\xi}_1) = (\bar{\xi}_1 \otimes 1 + 1 \otimes \bar{\xi}_1)^{2k+1}(1 \otimes \sigma\bar{\xi}_1) + \bar{\xi}_1^{2k+1} \otimes \bar{\xi}_1^2 + \bar{\xi}_2 \otimes \bar{\xi}_1^{2k} + \bar{\xi}_1^{2k+3} \otimes 1 + \bar{\xi}_1^{2k}\bar{\xi}_2 \otimes 1.$$

In the case of $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$, adding σ-cycles of the form $\bar{\xi}_1^j$ does not affect the comodule structure on the source up to a change of basis, since $\bar{\xi}_1^j$ is also in the target. If we add a σ-cycle in $H_*(THH(H\mathbb{F}_2))$ that is not in the source, then this affects the comodule structure on the source. We therefore compute the map up to this bit of indeterminacy. In summary, $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$ maps to $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$ up to σ-cycles in $H_*(THH(H\mathbb{F}_2))$ that are not in the image of $H_*(THH(y(1)))$. In Proposition 3.12, we will describe the map ϕ_{n*} up to the same type of indeterminacy.

Remark 3.10. In fact, we can actually avoid indeterminacy in the previous example because $\bar{\xi}_1^{2k}\sigma\bar{\xi}_1$ is a σ-boundary. Since we know that $\bar{\xi}_1^{2k+1}$ maps to the element of the same name, we see that $\bar{\xi}_1^{2k}\sigma\bar{\xi}_k$ must map to the element of the same name without any indeterminacy. This argument no longer applies when studying ϕ_{n*} for $n \geq 2$ since there will typically be additional elements in lower Bökstedt filtration.

Definition 3.11. We write $\mathrm{bfilt}(x)$ for the Bökstedt filtration of an element x. Given the map

$$\phi_{n*} : H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$$

we define

$$J_n = (x : x \in H_*(THH(H\mathbb{F}_2)) - \operatorname{im}\phi_{n*}, \mathrm{bfilt}(x) \leq n \text{ and } \sigma(x) = 0)$$

to be the ideal generated by all σ-cycles x in $H_*(THH(H\mathbb{F}_2))$ in Bökstedt filtration less than or equal to n that are not in the image of ϕ_{n*}. We refer to these elements as the *complementary σ-cycles* in the proof of the following proposition.

In Proposition 3.12, we assume a change of basis that makes $\sigma\bar{\xi}_i$ a comodule primitive for all i.

Proposition 3.12. *Let $x_i = \sigma\bar{\xi}_i\sigma\bar{\xi}_{i+1}\dots\sigma\bar{\xi}_n$ The map*

$$\phi_{n*}\colon\ H_*(THH(y(n))) \to H_*(THH(H\mathbb{F}_2))$$

is determined by

$$\phi_{n*}(\bar{\xi}_i x_i) = \bar{\xi}_i x_i + \bar{\xi}_{n+1} \tag{3}$$

and for $y \in H_*(y(n))$ or $y \in E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots, \sigma\bar{\xi}_n)$

$$\phi_{n*}(y) = y. \tag{4}$$

For all remaining products of elements, the map ϕ_{n*} behaves as if it were multiplicative modulo complementary σ-cycles.

Proof. We begin with the proof of 3. The elements $\bar{\xi}_i x_i$ are σ-cycles in $H_*(THH(y(n)))$, but in $H_*(THH(H\mathbb{F}_2))$ we know that $\sigma(\bar{\xi}_i x_i) = \sigma\bar{\xi}_{n+1}$. Since $\bar{\xi}_n\sigma\bar{\xi}_n$ maps to $\bar{\xi}_n\sigma\bar{\xi}_n + z$ for z in lower Bökstedt filtration, we know that $\sigma(\bar{\xi}_i x_i + z) = 0$ and therefore that $\sigma(z) = \sigma\bar{\xi}_{n+1}$. In this case, the only element z in lower Bökstedt filtration such that $\sigma z = \sigma\bar{\xi}_{n+1}$ is $\bar{\xi}_{n+1}$ itself. We now proceed by downward induction on i to show $\bar{\xi}_i x_i$ maps to $\bar{\xi}_i x_i + \bar{\xi}_{n+1}$ for all $i \leq n$. By our inductive hypothesis, $\bar{\xi}_j x_j$ maps to $\bar{\xi}_j x_j + \bar{\xi}_{n+1}$ up to elements in lower Bökstedt filtration for all $j > i$. Now, $\bar{\xi}_i x_i$ maps to $\bar{\xi}_i x_i + z$ where either $z = \bar{\xi}_{n+1}$ or

$$z \in \{\bar{\xi}_n x_n, \bar{\xi}_{n-1}x_{n-1}, \dots, \bar{\xi}_{i+1}x_{i+1}\}$$

up to σ-cycles in $H_*(THH(y(n)))$. If the former holds, then we are done. If the latter holds, then we can add z to the source and we know that $\bar{\xi}_i x_i + z$ maps to $\bar{\xi}_i x_i + \bar{\xi}_{n+1}$ by the inductive hypothesis. Thus we have proven that 3 holds.

We now turn to 4. For $y \in H_*(y(n)))$ it is clear that $\phi_{n*}(y) = y$ because all such y are in Bökstedt filtration zero. For $y \in E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots, \sigma\bar{\xi}_n)$, we know that, after a possible change of basis, each of these elements y is a comodule primitive in $H_*(THH(y(n)))$ and therefore each such y maps to the element of the same name in $H_*(THH(H\mathbb{F}_2))$.

We now prove the last sentence of the proposition. We first begin with products of the form $y\bar{\xi}_i x_i$.

Case 1: If y is a σ-cycle, then

$$\phi_{n*}(y\bar{\xi}_i x_i) = y\bar{\xi}_i x_i + y\bar{\xi}_{n+1} \tag{5}$$

by the same proof as the one given for $\bar{\xi}_i x_i$. Note that all σ-cycles y are in even degree so $y\bar{\xi}_1 x_i$ must be in an odd degree. We therefore know that formula (5) holds.

Case 2: Suppose $y \in H_*(THH(y(n)))$ is not a σ-cycle and therefore $\sigma(y\bar{\xi}_i x_i) = \sigma(y)\bar{\xi}_i x_i$ in $H_*(THH(y(n)))$. Since $\sigma(y\bar{\xi}_i x_i) = \sigma(y)\bar{\xi}_i x_i + y\sigma\bar{\xi}_{n+1}$ in $H_*(THH(H\mathbb{F}_2))$, we must add a correcting term z in the target such that $\sigma(z) = y\sigma\bar{\xi}_{n+1}$. The only possibility is an element of the form $y\bar{\xi}_{n+1} + c$ where c is a σ-cycle. We note that this seems to add additional terms that cause a further discrepancy since $\sigma(y\bar{\xi}_{n+1}) = \sigma(y)\bar{\xi}_{n+1} + y\sigma\bar{\xi}_{n+1}$, but this extra term is accounted for since $\sigma(y)\bar{\xi}_i x_i$ is also of the form we are currently handling in Case 2 and thus $\sigma(y)\bar{\xi}_i x_i$ maps to $\sigma(y)\bar{\xi}_i x_i + \sigma(y)\bar{\xi}_n$. Thus, we also have that $y\bar{\xi}_i x_i$ maps to $y\bar{\xi}_i x_i + y\bar{\xi}_n + c$ where c is a σ-cycle. We can restrict to $c \in J_n$ by changing the element in the source of this map by a change of basis where we add on terms in lower Bökstedt filtration.

This covers all products that are divisible by $\bar{\xi}_i x_i$. Let $y \in H_*(y(n))$ and let

$$w \in E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots \sigma\bar{\xi}_n).$$

Case 1: Suppose y, and consequently yw, is a σ-cycle. Then yw maps to $yw + c$ where c is a σ-cycle. By the same argument as earlier, we may assume $c \in J_n$. This completes this case.

Case 2: Suppose y is not a σ-cycle. Then $\sigma(yw) = \sigma(y)w \neq 0$ in $H_*(THH(y(n)))$, but $\sigma(yw) = \sigma(y)w$ in $H_*(THH(H\mathbb{F}_2))$. Therefore, if yw maps to the class of the same name, then the map is well defined as a map of $E(\sigma)$-modules. Therefore we may only possibly add σ-cycles to yw in the target, and as above, we only need to consider complementary σ-cycles. $\square$

We now note that the map ϕ_{n*} is also a map of $E(\sigma)$-modules in $\mathcal{A}_*$-comodules. Since the map is exotic in some cases, it produces an exotic $\mathcal{A}_*$-coaction on these elements. This will be described in detail in the following corollary.

Corollary 3.13. *The $\mathcal{A}_*$-coaction on $H_*(THH(y(n)))$ is determined by the formula*

$$\nu_n(\bar{\xi}_i x_i) = 1 \otimes \bar{\xi}_i x_i + \sum_{0<j+k=i} \bar{\xi}_j \otimes \bar{\xi}_k^{2^j} x_i + \bar{\xi}_i \otimes x_i + \sum_{j+k=n+1} \bar{\xi}_j \otimes \bar{\xi}_k^{2^{n+1}} \tag{6}$$

for $1 \leq i \leq n$ the usual coaction on $H_*(y(n))$ and the coaction on $E(\sigma\bar{\xi}_1, \dots \sigma\bar{\xi}_n)$ where each element in primitive, modulo possible additional terms in the coaction of products coming from added complementary σ-cycles. There is no indeterminacy in the formula (6).

Remark 3.14. Ultimately we will only be concerned with the map ϕ_{n*} as a map of $\mathcal{E}_*$-comodules and we will not use the $\mathcal{A}_*$-comodule structure. When we consider this map only as a map of $\mathcal{E}_*$-comodules there is no indeterminacy in the formulas for the coproduct above since all of the complementary σ-cycles have trivial $\mathcal{E}_*$-coaction.

4. Topological periodic cyclic homology of $y(n)$

In many classical trace methods computations, topological periodic cyclic homology is understood using the homotopical Tate spectral sequence described by Greenlees-May [27]. In Section 4.1, we explain why this method of understanding $TP(R)$ is not tractable when $R = y(n)$ for $n < \infty$. In Section 4.2, we apply an alternative approach to understanding $TP(R)$, the *homological* Tate spectral sequence, which was pioneered by Bruner-Rognes [21]. We analyze this spectral sequence to compute the *continuous* homology $H_*^c(TP(y(n)))$ in Proposition 4.6. We will elaborate on the meaning of the word "continuous" in this context when we recall this spectral sequence in the next section.

In Section 4.3, we construct the "localized inverse-limit Adams spectral sequence" and discuss its convergence. In Section 4.4, we use it to show that the continuous Morava K-theory $K(m)_*^c(TP(y(n)))$ vanishes for $1 \leq m \leq n$. In Section 4.5, we check that the map $TP(y(n)) \to TP(H\mathbb{F}_2)$ is a rational isomorphism. Assuming Conjecture 6.1, we find that the relative topological periodic cyclic homology $TP(y(n), H\mathbb{F}_2)$ has chromatic complexity at least $n+1$.

4.1. Limitations of the homotopical Tate spectral sequence.

Let R be an E_1-ring spectrum. The topological periodic cyclic homology spectrum $TP(R)$ arises in many classical trace methods computations. For example, when $p > 2$ is an odd prime, the spectrum $TP(H\mathbb{F}_p)$ appears in Hesselholt and Madsen's computation of the algebraic K-theory of finite algebras over the Witt vectors of perfect fields [29]. Similarly, it plays an important role in the computation of $TC(\mathbb{Z}_2; \mathbb{Z}/2)$ by Rognes [46]. In both cases, they analyze the mod p homotopical Tate spectral sequence

$$\widehat{E}^2 = \hat{H}^{-*}(\mathbb{T}; \pi_*(THH(R)); \mathbb{Z}/p)) \Rightarrow \pi_*(TP(R); \mathbb{Z}/p)$$

defined in [27]. We will review the filtration used to define this spectral sequence when we define the homological Tate spectral sequence in Subsection 4.2.

When $R = y(\infty) = H\mathbb{F}_p$, this spectral sequence is fairly straightforward. By Böksedt periodicity [19], $\pi_*(THH(H\mathbb{F}_p)) \cong P(\sigma\bar{\tau}_0)$ with $|\sigma\bar{\tau}_0| = 2$, so one has a familiar checkerboard pattern on the E^2-page and the spectral sequence collapses. On the other hand, when $R = y(n)$ for $n < \infty$, this spectral sequence is intractable.

Example 4.1. When $n = 0$ we have $y(0) = S$ and $THH(S) \simeq S$ as $\mathbb{T}$-spectra. There is an equivalence of spectra

$$TP(S) \simeq \Sigma^2 \mathbb{C}P^{\infty}_{-\infty}$$

by [27, Thm. 16.1], and the homotopy groups of $\mathbb{C}P^{\infty}_{-\infty}$ are even less well understood than the homotopy groups of spheres.

We have an identification of $THH(y(n))$, by Blumberg-Cohen-Schlichtkrull [16, Thm. 1], as

$$THH(y(n)) \simeq T(L^{\eta}(Bf))$$

where $f : \Omega J_{2^n-1}(S^2) \to BGL_1S$ is the map defining $y(n)$ as $T(f) = y(n)$ and $T(L^{\eta}(Bf))$ is the Thom spectrum of the composite map $L^{\eta}(Bf)$ defined as

$$LB\Omega J_{2^n-1}(S^2) \xrightarrow{L(Bf)} LB^2F \simeq BGL_1S \times B^2GL_1S \xrightarrow{BGL_1S\times\eta} BG_1S \times BGL_1S \to BGL_1S.$$

This spectrum has homotopy groups at least as complicated as $\pi_*(y(n))$, which are only known in a finite range. Since we want to understand large-scale phenomena in these homotopy groups, we will use a different approach.

4.2. **Homological $\mathbb{T}$-Tate spectral sequence for $THH(y(n))$.** In a note by Rognes [49], it is shown using the homological homotopy fixed point spectral sequence and the inverse-limit Adams spectral sequence [35] that there is an isomorphism of graded abelian groups

$$\pi_*(TC^-(H\mathbb{F}_2)) \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}\mathbb{Z}_2.$$

A similar argument shows that there is an isomorphism of graded abelian groups

$$\pi_*(TP(H\mathbb{F}_2)) \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}\mathbb{Z}_2.$$

Our goal in this section is to obtain similar results for $TP(y(n))$.

Definition 4.2 (Homological Tate spectral sequence [21]). Let R be an E_1-ring spectrum. The *homological Tate spectral sequence* has the form

$$\hat{E}^2 = \hat{H}^{-*}(\mathbb{T}; H_*(THH(R))) \Rightarrow H^c_*(TP(R)).$$

It arises from the Greenlees filtration of $THH(R)^{t\mathbb{T}} = [F(E\mathbb{T}_+, THH(R)) \wedge \widetilde{E\mathbb{T}}]^{\mathbb{T}}$ defined for $i \geq 0$ by setting (cf. [21, Sec. 2])

$$TP(R)[i] := [F(E\mathbb{T}_+, THH(R)) \wedge \widetilde{E\mathbb{T}}/\widetilde{E\mathbb{T}}_i]^{\mathbb{T}}$$

where $\widetilde{E\mathbb{T}}_\bullet$ is the skeletal filtration of $\widetilde{E\mathbb{T}}$ defined in [26, p.46]. The inverse-limit

$$H^c_*(TP(R)) := \lim_{\substack{\leftarrow \\ i}} H_*(TP(R)[i])$$

is called the *continuous homology* of $TP(R)$. For $0 \leq n \leq \infty$, we will denote the E^r-page of the homological Tate spectral sequence converging to $H^c_*(TP(y(n)))$ by $\hat{E}^r(n)$.

We have the following identification of $\hat{E}^2(n)$.

Lemma 4.3. *There is an additive isomorphism*

$$\hat{E}^2(n) \cong P(t, t^{-1}) \otimes H_*(THH(y(n))) \cong P(t, t^{-1}) \otimes P(\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_n)$$

where $|t| = (-2, 0)$, $|\bar{\xi}_i| = (0, 2^i - 1)$, *and* $|\sigma\bar{\xi}_i| = (0, 2^i)$.

The differentials in the homological Tate spectral sequence were studied by Bruner and Rognes in [21, Sec. 3].

Lemma 4.4. [21, Prop. 3.2] *Suppose η is trivial in $THH_*(R)$. The d^2-differentials in the homological Tate spectral sequence are induced by the circle action $\sigma\colon\ \mathbb{T}\wedge R \to \mathbb{T}_+ \wedge THH(R) \to THH(R)$, i.e. if $x \in \hat{E}^2_{**}$, then $d^2(x) = t \cdot \sigma(x)$ where $t \in \hat{E}^2_{2,0}$ corresponds to the generator of*

$$H^*(\mathbb{T}; \mathbb{F}_2) \cong H^*(\mathbb{C}P^{\infty}; \mathbb{F}_2) \cong P(t).$$

Therefore in order to compute $\hat{E}^3(n)$, we need to understand the $\mathbb{T}$-action on $THH(y(n))$. This can be understood using Proposition 3.12 and Lemma 3.4.

Definition 4.5. [21, Prop. 6.1.(a)] Let $k \geq 1$. Define $\bar{\xi}'_{k+1} \in H_*(THH(H\mathbb{F}_2))$ by

$$\bar{\xi}'_{k+1} := \bar{\xi}_{k+1} + \bar{\xi}_k \sigma \bar{\xi}_k.$$

Proposition 4.6. *There is an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$H^c_*(TP(y(n))) \cong P(t, t^{-1}) \otimes P(\bar{\xi}_1^2, \dots, \bar{\xi}_n^2) \otimes E(\bar{\xi}'_2, \dots, \bar{\xi}'_n) \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n)$$

with $|t| = (-2, 0)$, $|\bar{\xi}_i| = (0, 2^i - 1)$, and $|\sigma \bar{\xi}_i| = (0, 2^i)$.

Proof. First, $\hat{E}^2_{**}(n)$ was computed in Lemma 4.3. Although this may not a multiplicative spectral sequence (since $THH(y(n))$ may not be a ring spectrum), it is a module over the spectral sequence for the sphere, which in our notation is $\hat{E}^2_{*,*}(0)$. Consequently, $d^r(t) = 0$ and the differentials are t-linear.

We have differentials $d^2(\bar{\xi}_k) = t\sigma\bar{\xi}_k$. By t-linearity, we obtain $d^2(t^m \bar{\xi}_k) = t^{m+1}\sigma\bar{\xi}_k$ for $m \in \mathbb{Z}$. Further, $d^r(t) = 0$ for all r by comparison to the homological Tate spectral sequence for the sphere spectrum.

Recall the notation $x_i = \sigma\bar{\xi}_i \cdots \sigma\bar{\xi}_n$. We observe that any class of the form $y\bar{\xi}_i x_i$, where y is a σ-cycle, is a d^2-cycle in the homological Tate spectral sequence converging to $H^c_*(THH(y(n))^{t\mathbb{T}})$. However, many of these classes are also d^2-homologous. In particular, we have

$$d^2(y\bar{\xi}_i\bar{\xi}_n\sigma\bar{\xi}_i \cdots \sigma\bar{\xi}_{n-1})) = ty\bar{\xi}_i x_i + ty\bar{\xi}_n x_n.$$

Using these relations and the fact that this spectral sequence is a module over the spectral sequence for the sphere, we obtain an additive isomorphism (cf. [21, Proposition 6.1])

$$\hat{E}^3_{**}(n) \cong P(t, t^{-1}) \otimes P(\bar{\xi}_1^2, \dots, \bar{\xi}_n^2) \otimes P(\bar{\xi}'_2, \bar{\xi}'_3, \dots, \bar{\xi}'_n) \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n).$$

To see that there are no further differentials, we use the map of spectral sequences induced by the $\mathbb{T}$-equivariant map $THH(y(n)) \to THH(H\mathbb{F}_2)$. The homological Tate spectral sequence converging to $H^c_*(THH(H\mathbb{F}_2)^{t\mathbb{T}})$, the case $n = \infty$, has $\hat{E}^3$-page

$$\hat{E}^3_{**}(\infty) \cong P(t, t^{-1}) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2^2, \dots) \otimes E(\bar{\xi}'_2, \bar{\xi}'_3, \dots).$$

All of the generators are permanent cycles by [21, Thm. 5.1], so there are no further differentials. The map $\hat{E}^3_{*,*}(n) \to \hat{E}^3_{*,*}(\infty)$ is injective by Proposition 3.12 so we can conclude that there is also an isomorphism $\hat{E}^3(n) \cong \hat{E}^\infty(n)$. □

A similar proof can be used to compute the homology of the spectra $TP(y(n))[i]$ which were used to define the filtration of $TP(y(n))$ giving rise to the homological Tate spectral sequence.

Corollary 4.7. *There is an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$\begin{aligned} H_*(TP(y(n))[i]) \cong\, & P(t^{-1})\{t^{i-1}\} \otimes P(\bar{\xi}_1^2, \dots, \bar{\xi}_n^2) \otimes E(\bar{\xi}'_2, \dots, \bar{\xi}'_n) \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n) \\ & \oplus P(\bar{\xi}_1, \dots, \bar{\xi}_n)\{t^i\} \end{aligned}$$

with $|t| = (-2, 0)$, $|\bar{\xi}_i| = (0, 2^i - 1)$, $|\sigma\bar{\xi}_i| = (0, 2^i)$, and $P(t^{-1})\{t^{i-1}\}$ is viewed as a $P(t^{-1})$-submodule of $P(t, t^{-1})$.

4.3. **Localized and inverse-limit Adams spectral sequences.** Following the computation of $\pi_*(TC^-(H\mathbb{F}_2))$ by Rognes [49], we would like to use the inverse-limit Adams spectral sequence to compute the homotopy groups $\pi_*(TP(y(n)))$. This spectral sequence is discussed in detail in [36, Sec. 2]. For now, it suffices to say that if $X = \lim_i X_i$ is the homotopy limit of bounded below spectra X_i of finite type, then the inverse-limit Adams spectral sequence arises from the filtration of X obtained by taking the inverse-limit of compatible Adams filtrations of the spectra X_i. This spectral sequence has the form

$$E_2^{*,*} = \mathrm{Ext}^{**}_{\mathcal{A}_*}(\mathbb{F}_2, H^c_*(X)) \Rightarrow \pi_*(X)$$

where the left-hand side is computed using the continuous $\mathcal{A}_*$-coaction on $H_*^c(X)$.

We cannot expect to use this spectral sequence to compute the stable homotopy groups $TP_*(y(n)) = \pi_*(TP(y(n)))$ for any $n < \infty$, however. The main difficulty stems from the fact that $\mathcal{A}_*$ coacts nontrivially on $P(t, t^{-1}) \subset H_*^c(TP(y(n)))$.

Remark 4.8. This problem is avoided when $n = \infty$ as follows. There is an $\mathcal{A}_*$-comodule isomorphism

$$H_*^c(TP(H\mathbb{F}_2)) \cong P(t, t^{-1}) \otimes H_*(H\mathbb{Z}_2) \cong P(t, t^{-1}) \otimes (\mathcal{A}//E(0))_*.$$

Change-of-rings then simplifies the E_2-page to

$$E_2^{*,*} \cong Ext^{*,*}_{E(\bar{\xi}_1)}(\mathbb{F}_2, P(t, t^{-1})).$$

Since $\bar{\xi}_1$ is in an odd degree and $P(t, t^{-1})$ is concentrated in even degrees, the $E(\bar{\xi}_1)$-coaction on $P(t, t^{-1})$ is trivial. Therefore the E_2-page simplifies further to

$$E_2^{*,*} \cong P(t, t^{-1}) \otimes Ext^{*,*}_{E(\bar{\xi}_1)}(\mathbb{F}_2, \mathbb{F}_2)$$

and so the spectral sequence collapses for degree reasons.

More generally, the $\mathcal{A}_*$-coaction restricted to sub-Hopf-algebras generated by elements in odd degrees, such as $E(n)$, is trivial on $P(t, t^{-1})$. Recall from Section 2 that there are isomorphisms

$$H^*(k(m)) \cong A//E(Q_m),$$
$$H_*(k(m)) \cong \mathcal{A}_* \square_{E(\bar{\xi}_{m+1})} \mathbb{F}_2.$$

Since $|\bar{\xi}_{m+1}|$ is odd, the $E(\bar{\xi}_{m+1})$-coaction on $P(t, t^{-1})$ is trivial. Equivalently, the $E(Q_m)$-action on $P(t, t^{-1})$ is trivial. Therefore we can potentially compute $k(m)_*(TP(y(n)))$. The following lemma provides a change-of-rings isomorphism for the inverse-limit Adams spectral sequence.

Lemma 4.9. *Suppose $X = \lim X_i$ is the homotopy inverse-limit of a tower of spectra X_i which are all bounded below and of finite type over $\mathbb{F}_p$. Then there is an inverse-limit Adams spectral sequences of the form*

$$E_2^{**} = \mathrm{Ext}^*_{E(Q_m)}(H_c^*(X), \mathbb{F}_2) \Rightarrow k(m)_*^c(X)$$

which converges strongly to the continuous connective Morava K-theory of X. Dually, there is an inverse-limit Adams spectral sequences of the form

$$E_2^{**} = \mathrm{Cotor}^*_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*^c(X)) \Rightarrow k(m)_*^c(X).$$

Proof. Setting $Y_i := k(m) \wedge X_i$ defines a tower of spectra which are still bounded below and of finite type over $\mathbb{F}_p$. Let $Y = \lim Y_i$. The desired spectral sequence follows from [36, Prop. 2.2], except that their E_2-page would have the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(\operatorname*{colim}_{i \to -\infty} H^*(k(m) \wedge X_i), \mathbb{F}_2) \Rightarrow \pi_{t-s}(\widehat{Y}_2).$$

The proof of [36, Prop. 2.2] shows that the first E_2-term above can be obtained as the inverse-limit of the E_2-terms of the Adams spectral sequences

$$E_2^{**}(i) = \mathrm{Ext}^{**}_{\mathcal{A}}(H^*(Y_i), \mathbb{F}_2) \cong \mathrm{Ext}^{**}_{\mathcal{A}}(\mathcal{A}/E(Q_m) \otimes H^*(Y_i), \mathbb{F}_2) \cong \mathrm{Ext}^{**}_{E(Q_m)}(H^*(Y_i), \mathbb{F}_2).$$

Since

$$\lim_i \mathrm{Hom}_{E(Q_m)}(H^*(Y_i), N) \cong \mathrm{Hom}_{E(Q_m)}(\operatorname*{colim}_i H^*(Y_i), N)$$

for $N = \Sigma^t \mathbb{F}_2$, we have the desired E_2-term. □

In order to determine the chromatic complexity of $TP(y(n))$, we need to compute its Morava K-theory $K(m)_* TP(y(n))$ for $m \leq n + 1$. We emphasize that the previous proof relies on the bounded below condition, so in particular, we cannot use the inverse-limit Adams spectral sequence to compute the Morava K-theory $K(m)_*(TP(y(n)))$ since $K(m)_* \cong \mathbb{F}_2[v_m^{\pm 1}]$ is not bounded below. We were able to use the localized Adams spectral sequence in Section 2 to compute the Morava

$$\begin{tikzcd}
\lim\limits_{i\to\infty} k(m)\wedge TP[i] \arrow[r, "v_m\wedge 1"] \arrow[d] & \lim\limits_{i\to\infty} \Sigma^{2p-2}k(m)\wedge TP[i] \arrow[r, "v_m\wedge 1"] \arrow[d] & \cdots \arrow[r] & ? \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
k(m)\wedge TP[i+1] \arrow[r, "v_m\wedge 1"] \arrow[d] & \Sigma^{2p-2}k(m)\wedge TP[i+1] \arrow[r, "v_m\wedge 1"] \arrow[d] & \cdots \arrow[r] & k(m)\widehat{\wedge TP}[i+1] \arrow[d] \\
k(m)\wedge TP[i] \arrow[r, "v_m\wedge 1"] \arrow[d] & \Sigma^{2p-2}k(m)\wedge TP[i] \arrow[r, "v_m\wedge 1"] \arrow[d] & \cdots \arrow[r] & k(\widehat{m)\wedge T}P[i] \arrow[d] \\
\vdots & \vdots & & \vdots
\end{tikzcd}$$

FIGURE 1. Inverse and directed systems for Morava K-theory of TP

K-theory $K(m)_*(TP(y(n)))$, but in this case we need to to understand how the localized Adams resolutions used to compute $K(m)_*(TP(y(n))[i])$ behave under inverse-limits. The situation is illustrated in Figure 4.3, where we use the shorthand TP and $TP[i]$ for $TP(y(n))$ and $TP(y(n))[i]$.

Associated to each row is a localized Adams spectral sequence of the form

$$v_m^{-1}E_2 = v_m^{-1}Ext_{E(Q_m)}(\mathbb{F}_2, H_*(TP[i])) \Rightarrow \pi_*(k(\widehat{m)\wedge T}P[i]).$$

Convergence of this spectral sequence is not immediate. To state the convergence theorem we will use, we recall some results from [38, Sec. 2.3]. Let

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

be an Adams resolution of X. Let $f : X \to \Sigma^{-d}X$ be the v_n-map which we will iterate to define the telescope $\widehat{X}$ as above. Suppose there is a lifting $\tilde{f} : X \to \Sigma^{-d}X_{s_0}$ for some $s_0 \geq 0$. This lifting induces maps $\tilde{f} : X_s \to X_{s+s_0}$ for $s \geq 0$, i.e. for each X_s in the Adams resolution above. We iterate these maps to define the telescopes $\widehat{X_s}$. Setting $X_s = X$ for $s < 0$ produces a tower

$$\cdots \leftarrow \widehat{X_{-1}} \leftarrow \widehat{X_0} \leftarrow \widehat{X_1} \leftarrow \cdots .$$

The resulting full plane spectral sequence is the localized Adams spectral sequence.

Theorem 4.10. [38, Thm. 2.13] *For a spectrum X equipped with maps f and $\tilde{f}$ as above, in the localized Adams spectral sequence for $\pi_*(\hat{X})$ we have*

- *The homotopy direct limit $\lim\limits_{s\to\infty} \hat{X}_{-s}$ is the telescope $\hat{X}$.*
- *The homotopy inverse-limit $\lim\limits_{s\to\infty} \hat{X}_s$ is contractible if the original (unlocalized) Adams spectral sequence has a vanishing line of slope s_0/d at E_r for some finite r, i.e. if there are constants c and r such that*

$$E_r^{s,t} = 0 \quad \textit{for} \quad s > c + (t-s)(s_0/d).$$

(In this case, we say f has a parallel lifting *$\tilde{f}$.)*
- *If f has a parallel lifting, this localized Adams spectral sequence converges to $\pi_*(\tilde{X})$.*

Mahowald, Ravenel, and Shick compute $v_n^{-1}E_2$ for the spectra $y(n)$ in [38, Sec. 2.3]. Their computations show that the localized Adams spectral sequence for $\pi_*(\widehat{y(n)})$ converges. The same argument does not apply to show convergence for the localized Adams spectral sequence for $\pi_*(\widehat{z(n)})$

since $\mathbb{Z}_2 \subset \pi_{2^{n+1}-2}(z(n))$ implies that there is no vanishing line in the unlocalized Adams spectral sequence. However, this theorem can be applied to $k(m) \wedge (z(n)/v_n)$ when $1 \leq m \leq n$.

Lemma 4.11. *The localized Adams spectral sequence for $K(m)_*(z(n)/v_n)$ converges strongly for $1 \leq m \leq n$.*

Proof. By Corollary 2.19 the Margolis homology groups $H(z(n)/v_n; Q_m)$ vanish for $1 \leq m \leq n$, so the E_2-page of the unlocalized Adams spectral sequence for $k(m)_*(z(n)/v_n)$ is concentrated in Adams filtration zero, i.e.

$$Ext^{i,*}_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*(z(n)/v_n)) = 0$$

for $i > 0$. Therefore the unlocalized Adams spectral sequence for $k(m) \wedge z(n)/v_n$ has a horizontal vanishing line and the previous theorem applies for $1 \leq m \leq n$. $\square$

We would like a similar convergence result for the rows in the above diagram. To do so, we need to understand the E_2-page of the unlocalized Adams spectral sequence converging to $\pi_*(k(m) \wedge TP(y(n))[i])$ for $i \in \mathbb{Z}$ and $1 \leq m \leq n$. In particular, we need to understand the $\mathcal{A}_*$-coaction on $H_*(TP(y(n)[i])$ for $i \in \mathbb{Z}$. This can be computed from the continuous $\mathcal{A}_*$-coaction on $H^c_*(TP(y(n))$, which can be computed using the fact that the map in continuous homology

$$\phi_{n*}\colon\ H^c_*(TP(y(n))) \to H^c_*(TP(H\mathbb{F}_2))$$

induced by the map $y(n) \to H\mathbb{F}_2$ is a map of continuous $\mathcal{A}_*$-comodules.

Lemma 4.12. *The continuous $\mathcal{A}_*$-coaction*

$$\nu_n : H^c_*(TP(y(n))) \to \mathcal{A}_* \widehat{\otimes} H^c_*(TP(y(n)))$$

satisfies

$$\phi_{n*}(\nu_n(x)) = \nu_\infty(\phi_{n*}(x))$$

where ϕ_{n*} is the map in continuous homology induced by $\phi_n : TP(y(n)) \to TP(H\mathbb{F}_2)$ and ν_∞ is the $\mathcal{A}_*$-coaction on $H^c_*(TP(H\mathbb{F}_2))$.

We can identify a piece of the continuous homology $H^c_*(TP(y(n)))$ computed in Proposition 4.6 with the homology $H_*(z(n)/v_n)$. Our identification may not hold as $\mathcal{A}_*$-comodules due to the possibility of complementary σ-cycles in the image of ϕ_{n*}, but it is compatible with the action of certain Steenrod operations. Recall the definition of $\mathcal{E}$ from Definition 2.21

Lemma 4.13. *There is an isomorphism of continuous $\mathcal{E}_*$-comodules*

$$H^c_*(TP(y(n))) \cong P(t^{\pm 1}) \otimes H_*(z(n)/v_n).$$

Proof. First, note that the E-action on $P(t^{\pm 1})$ is trivial because $|t| = -2$ and each Q_i is in an odd degree for all $i \geq 0$. The desired isomorphism is therefore given by a map

$$P(\bar{\xi}_1^2, \dots, \bar{\xi}_n^2) \otimes E(\bar{\xi}_2', \dots, \bar{\xi}_n') \otimes E(\bar{\xi}_n \sigma \bar{\xi}_n) \to H_*(z(n)/v_n)$$

which is determined by

$$\bar{\xi}_i^2 \mapsto \bar{\xi}_i^2, 1 \leq i \leq n, \quad \bar{\xi}_i' \mapsto \bar{\xi}_i, 2 \leq i \leq n, \quad \bar{\xi}_n \sigma \bar{\xi}_n \mapsto \bar{\xi}_{n+1}.$$

This is clearly an additive isomorphism, so it only remains to calculate the action of Q_i.

By Lemma 4.12, we can compute the continuous $\mathcal{A}_*$-coaction

$$\nu_n\colon\ H^c_*(TP(y(n)) \to \mathcal{A}_* \widehat{\otimes} H^c_*(TP(y(n))$$

by comparison with the known coaction

$$\nu_\infty\colon\ H^c_*(TP(H\mathbb{F}_2)) \to \mathcal{A}_* \widehat{\otimes} H^c_*(TP(H\mathbb{F}_2)).$$

We have $\phi_{n*}(x) = x$ for $x \in P(\xi_1^2, \dots, \xi_n^2)$ since $\mathrm{bfilt}(x) = 0$, so $\nu_n(x) = \nu_\infty(x)$. We then calculate $\nu_n(\bar{\xi}_i')$ as follows. We have $\phi_{n*}(\bar{\xi}_i') = \bar{\xi}_i' + y$ where y is a σ-cycle with $\mathrm{bfilt}(y) = 0$. If $y \notin J_n$, then we have $\nu_n(\bar{\xi}_i') = \nu_\infty(\bar{\xi}_i')$ up to the addition of elements in lower Bökstedt filtration. If $y \in J_n$, then y is divisible by $\bar{\xi}_{n+r}^2$ or $\sigma\bar{\xi}_{n+r}$ for $r \geq 1$. In both cases, $\nu_\infty(y)$ does not contain any terms of the form $\bar{\xi}_i \otimes z$ for any i, so Q_i acts trivially on y for all i and the action of Q_i on $\bar{\xi}_i'$ is unaffected by y. Finally, we have $\phi_{n*}(\bar{\xi}_n\sigma\bar{\xi}_n) = \bar{\xi}_n\sigma\bar{\xi}_n + \bar{\xi}_{n+1}$. Therefore $\nu_n(\bar{\xi}_n\sigma\bar{\xi}_n) = \nu_\infty(\bar{\xi}_n\sigma\bar{\xi}_n) + \bar{\xi}_{n+1} \otimes 1 + \cdots$ so $Q_i(\bar{\xi}_n\sigma\bar{\xi}_n) = Q_i(\bar{\xi}_{n+1}')$ for all i. The result follows. □

We immediately obtain the following identification.

Corollary 4.14. *For any $m \geq 0$, there is an isomorphism*

$$Ext_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*^c TP(y(n))) \cong Ext_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*(\prod_{i\in\mathbb{Z}} \Sigma^{2i} z(n)/v_n))$$

between the E_2-page of the inverse-limit Adams spectral sequence converging to $k(m)_*^c(TP(y(n)))$ and the Adams spectral sequence converging to $k(m)_*(\prod_{i\in\mathbb{Z}} \Sigma^{2i} z(n)/v_n)$.

We emphasize that this is just an isomorphism of E_2-pages of spectral sequences, and not an isomorphism of spectral sequences. We do not expect there to be an equivalence of spectra $TP(y(n)) \simeq \prod_{i\in\mathbb{Z}} \Sigma^{2i} z(n)/v_n$ since such an equivalence does not hold when $n = 0$ (where the expression $z(0)/v_0$ is interpreted as S since there is an isomorphism $\pi_*(z(n)/p) \cong \pi_*(y(n))$ and $v_0 = p$. However, if we replace $\prod_{i\in\mathbb{Z}} \Sigma^{2i} z(n)/v_n$ by $(z(n)/v_n)^{t\mathbb{T}}$, then there is an equivalence of spectra when $n = 0$ and $n = \infty$ with the convention that $v_\infty := 0$. We therefore make the following conjecture.

Conjecture 4.15. *There is an equivalence $TP(y(n)) \simeq (z(n)/v_n)^{t\mathbb{T}}$.*

Note that the main obstacle towards proving this conjecture is constructing a $\mathbb{T}$-equivariant map $z(n)/v_n \to THH(y(n))$. In the case $n = \infty$, one gets lucky because $K(H\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$ and therefore there is a map of cyclotomic spectra $H\mathbb{Z}_p^{\mathrm{triv}} \to THH(\mathbb{F}_p)$, as observed by Nikolaus-Scholze [43].

By essentially the same proof as that of 4.13, we have the following corollary;

Corollary 4.16. *There is an isomorphism of continuous E_*-comodules*

$$H_*^c(TP(y(n))[i]) \cong H_*(z(n)/v_n) \otimes P(t) \otimes \mathbb{F}_2\{t^{-i+1}\} \oplus H_*(y(n)) \otimes \mathbb{F}_2\{t^i\}.$$

Consequently, for the terms in the rows of Diagram 4.3, the isomorphism of E_2-pages takes the following form.

Corollary 4.17. *For any $m \geq 0$, there is an isomorphism*

$$Ext^*_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*(TP(y(n))[i]) \cong Ext^*_{E(\bar{\xi}_{m+1})}(\mathbb{F}_2, H_*(\prod_{j\geq -i+1} \Sigma^{2j} z(n)/v_n) \oplus H_*(\Sigma^{-2i} y(n)))$$

between the E_2-page of the Adams spectral sequence converging to $k(m)_*(TP(y(n))[i])$ and the Adams spectral sequence converging to $k(m)_*(\prod_{j\geq -i+1} \Sigma^{2j} z(n)/v_n \vee \Sigma^{-2i} y(n))$.

Note that this isomorphism is visible at the level of E_2-pages of unlocalized Adams spectral sequences. Therefore there is a horizontal vanishing line in the unlocalized Adams spectral sequence converging to $k(m)_*(TP(y(n))[i])$ which follows from the horizontal vanishing line in the unlocalized Adams spectral sequence for $k(m)_*(z(n)/v_n)$ and $k(m)_*(y(n))$

Corollary 4.18. *The localized Adams spectral sequence converges to $K(m)_*(TP(y(n))[i])$ for any $1 \leq m \leq n$ and $i \geq 0$.*

Therefore in Figure 4.3, we can associate to each row (except possibly the top row) for $i \in \mathbb{Z}$ a localized Adams spectral sequence which converges strongly to $K(m)_*(TP(y(n))[i])$. Further, we can associate to each column (except possibly the rightmost column) an inverse-limit Adams spectral sequence which converges strongly to $k(m)_*^c(TP(y(n)))$.

There are two possible ways to arrive at the top-right corner of Figure 4.3. We will focus on the result of taking the inverse-limit of the rightmost column, which is by definition the *continuous Morava K-theory*

$$K(m)^c_*(TP) := \lim_{i\to-\infty} K(m)_*(TP[i]).$$

Remark 4.19. Generalized homology does not generally commute with homotopy limits, for example, consider the sequence $\cdots \to S/p^3 \to S/p^2 \to S/p$. The limit of this sequence is S_p, so $H\mathbb{Q}_*S_p \cong \mathbb{Q}_p$ whereas $H\mathbb{Q}_*S/p^n \cong 0$ for all n so $\lim H\mathbb{Q}_*S/p^n \cong 0$. Nevertheless, work in progress of the first author and Salch [5] implies that continuous Morava K-theory and Morava K-theory agree in many cases. We will return to this is Section 6.

4.4. **Chromatic complexity of $TP(y(n))$.** Using the complexes from Section 2, we can compute the Margolis homology of $TP(y(n))[i]$.

Lemma 4.20. *The Margolis homology of $TP(y(n))[i]$ is given by*

$$H(TP(y(n))[i]; Q_m) \cong \begin{cases} P(t^{-1})\{t^{i+1}\} & \text{if } m = 0, \\ 0 & \text{if } 1 \le m \le n-1, \\ \Sigma^{-2i} H_*(y(n)) & \text{if } m = n, \\ H_*(TP(y(n))[i]) & \text{if } m \ge n+1. \end{cases}$$

Proof. We have an isomorphism of $\mathcal{E}_*$-comodules

$$H_*(TP(y(n))[i]) \cong P(t^{-1})\{t^i\} \otimes H_*(z(n)/v_n)$$

by Lemma 4.13, so we can compute $H(TP(y(n))[i]; Q_i)$ in terms of $H(z(n)/v_n; Q_i)$ for all i by Lemma 2.22. We have the relation $Q_n(t^i) = 0$ for all $i \in \mathbb{Z}$ for degree reasons. Therefore $H(P(t,t^{-1}); Q_m) \cong P(t,t^{-1})$ for all $m \ge 0$. The Künneth isomorphism and Corollary 2.19 then prove the lemma for $m \le n$. When $m \ge n+1$, the proof is similar to the proof in Corollary 2.19. ☐

We can now show vanishing for continuous Morava K-theory of $TP(y(n))$.

Theorem 4.21. *For $1 \le m \le n < \infty$, the continuous Morava K-theory of $TP(y(n))$ vanishes, i.e.*

$$K(m)^c_*(TP(y(n))) = 0.$$

Proof. Let $1 \le m \le n-1$ and let $X_i = K(m) \wedge TP(y(n))[i]$ be the tower at the right-hand side of Figure 4.3. Then by Corollary 4.18 and Lemma 4.20, we have $\pi_*(X_i) = 0$ for all $i \in \mathbb{Z}$. In particular, X_i is bounded below and of finite type over $\mathbb{F}_2$ for all $i \in \mathbb{Z}$. Applying [36, Prop. 2.2] gives a strongly convergent inverse-limit Adams spectral sequence

$$E_2^{**} = Ext^{**}_{\mathcal{A}}(\operatorname*{colim}_{i\to-\infty} H^*(K(m) \wedge TP(y(n))[i]), \mathbb{F}_2) \Rightarrow K(m)^c_*(TP(y(n)))$$

where we have made the identification

$$\pi_*(\lim X_i) = \pi_*(\lim K(m) \wedge TP(y(n))[i]) = K(m)^c_*(TP(y(n))).$$

Since $H^*(K(m) \wedge TP(y(n))[i]) = H^*(pt) = 0$, the E_2-term is zero.

Let $m = n$. Then the maps in the tower at the right-hand side of Figure 4.3 are all zero. Indeed, the nontrivial Margolis homology $H(TP(y(n))[i]; Q_n)$ is generated by classes $t^i x$ with $x \in H_*(THH(y(n)))$, so as i tends to $-\infty$, these classes become zero. Therefore $K(n)^c_*(TP(y(n))) \cong 0$.

Let $m = 0$. Then by Lemma 4.20 and the above discussion, the E_2-page of the localized inverse-limit Adams spectral sequence has the form

$$E_2^{**} = \lim_{i\to-\infty} P(t^{-1})\{t^i\} \otimes \mathbb{Q}_2 \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}\mathbb{Q}_2.$$

The spectral sequence collapses for degree reasons to give the desired isomorphism. ☐

Remark 4.22. An interesting case not covered by the Theorem 4.21 is the case $n = \infty$ and $m = 0$, i.e. the continuous rational homology of $TP(H\mathbb{F}_2)$. By [43, Cor. IV.4.8], the homotopy ring of $TP(H\mathbb{F}_2)$ is given by

$$\pi_*(TP(H\mathbb{F}_2)) \simeq \mathbb{Z}_2[t^{\pm 1}]$$

where $|t| = 2$. Therefore one has

$$K(0)_*(TP(H\mathbb{F}_2)) = H\mathbb{Q}_*(TP(H\mathbb{F}_2)) \cong \mathbb{Q}_2[t^{\pm 1}].$$

Usually one arrives at the above computation using the *homotopical* Tate spectral sequence, which collapses at E_2. Solving extensions in truncations of this spectral sequence is actually very subtle. Without truncating the spectral sequence at all, the copy of $\mathbb{Z}_2$ in $\pi_0(TP(H\mathbb{F}_2))$ is obtained by identifying the class $t\sigma\bar{\xi}_1 \in \hat{E}^{\infty}_{-2,2}$ with p. One then sees that p^i is detected by $t^i(\sigma\bar{\xi}_1)^i$.

However, the copy of $\mathbb{Z}_2$ in $\pi_0(TP(H\mathbb{F}_2))$ actually exists in the homotopy groups of the truncations $\pi_0(TP(H\mathbb{F}_2)[i])$. To see this, note that the mod two homology $H_*(TP(H\mathbb{F}_2)[i])$ is given by [49]

$$H_*(TP(H\mathbb{F}_2)[i]) \cong H_*(H\mathbb{Z})\{\dots, t^{-2}, t^{-1}, 1, t, \dots, t^{i-1}\} \oplus H_*(H\mathbb{F}_2)\{t^i\}.$$

The Adams spectral sequence collapses to show that $\pi_0(TP(H\mathbb{F}_2)[i]) \cong \mathbb{Z}_2$ for $i \geq 1$. We note that in order for this answer to be compatible with truncated Tate spectral sequence, there must be an elaborate pattern of hidden extensions in the truncated homotopical Tate spectral sequence converging to $\pi_*(TP(H\mathbb{F}_2)[i])$. In any case, we obtain an additive isomorphism

$$K(0)_*(TP(H\mathbb{F}_2)[i]) = H\mathbb{Q}_*(TP(H\mathbb{F}_2)[i]) \cong \bigoplus_{j \geq -i+1} \Sigma^{2j}\mathbb{Q}_2.$$

Taking the colimit over i, we obtain an additive isomorphism

$$K(0)^c_*(TP(H\mathbb{F}_2)) \cong \mathbb{Q}_2[t^{\pm 1}] \cong K(0)_*(TP(H\mathbb{F}_2)).$$

This provides an example where continuous Morava K-theory agrees with (just) Morava K-theory. It also provides some evidence that we can circumvent issues with multiplicative structure in the homotopical Tate spectral sequence by using the homological Tate spectral sequence followed by the Adams spectral sequence.

4.5. **Chromatic complexity of relative topological periodic cyclic homology.** In this subsection we will compute the continuous Morava K-theory of relative topological periodic cyclic homology. In the proof of the previous theorem, we computed $K(m)_*(TP(y(n))[i])$ for $0 \leq m \leq n+1$. We will need the following specialization to the case $n = \infty$.

Lemma 4.23. *The Morava K-theory of the truncations $TP(H\mathbb{F}_2)[i]$ is given by*

$$K(m)_*(TP(H\mathbb{F}_2)[i]) \cong \begin{cases} \bigoplus_{j \geq -i+1} \Sigma^{2j}\mathbb{Q}_2 & \text{if} \quad m = 0, \\ 0 & \text{if} \quad 1 \leq m \leq n. \end{cases}$$

Proof. We saw above that

$$H_*(TP(H\mathbb{F}_2)[i]) \cong H_*(H\mathbb{Z})\{\dots, t^{-2}, t^{-1}, 1, t, \dots, t^{i-1}\} \oplus H_*(H\mathbb{F}_2)\{t^i\}.$$

Applying Lemma 2.17 with $n = \infty$ shows that the Margolis homology of $H\mathbb{Z}$ is given by

$$H(H\mathbb{Z}; Q_m) \cong \begin{cases} \mathbb{F}_2 & \text{if} \quad m = 0, \\ 0 & \text{if} \quad m > 0. \end{cases}$$

As above, Q_m acts trivially on t^i for all i, so we have

$$H(TP(H\mathbb{F}_2)[i]; Q_m) \cong \begin{cases} P(t^{-1})\{t^{i+1}\} & \text{if} \quad m = 0, \\ 0 & \text{if} \quad m > 0. \end{cases}$$

These computations imply horizontal vanishing lines in the unlocalized Adams spectral sequences converging to $k(m)_*(TP(H\mathbb{F}_2))$ for $m > 0$ and therefore imply convergence of the localized Adams spectral sequence computing $K(m)_*(TP(H\mathbb{F}_2)[i])$ for $m > 0$. Therefore we obtain the desired vanishing when $m > 0$. The case $m = 0$ follows from our remarks at the end of Section 4.4. □

Theorem 4.24. *For $0 \leq m \leq n$, the m-th continuous Morava K-theory of $TP(y(n), H\mathbb{F}_2)$ vanishes, i.e. there is an isomorphism*

$$K(m)^c_*(TP(y(n), H\mathbb{F}_2)) \cong 0.$$

Proof. Clearly we have $TP(y(n), H\mathbb{F}_2) = \lim_{i\to\infty} TP(y(n), H\mathbb{F}_2)[i]$. For each i we obtain a long exact sequence in Morava K-theory

$$\begin{aligned} \cdots \to K(m)_*(TP(y(n), H\mathbb{F}_2)[i]) &\to K(m)_*(TP(y(n))[i]) \\ &\to K(m)_*(TP(H\mathbb{F}_2)[i]) \to K(m)_{*-1}(TP(y(n), H\mathbb{F}_2)[i]) \to \cdots \end{aligned}$$

When $m = 0$, we have shown

$$K(0)_*(TP(y(n))[i]) \cong \bigoplus_{j \geq -i+1} \Sigma^{2j}\mathbb{Q} \cong K(0)_*(TP(H\mathbb{F}_2)[i])$$

so we see that

$$K(0)_*(TP(y(n), H\mathbb{F}_2)[i]) \cong 0.$$

When $1 \leq m \leq n$, we have shown

$$K(m)_*(TP(y(n))[i]) \cong 0 \cong K(m)_*(TP(H\mathbb{F}_2)[i])$$

so we see that

$$K(m)_*(TP(y(n), H\mathbb{F}_2)[i]) \cong 0.$$

□

5. Topological negative cyclic homology of $y(n)$

In this section, we mimic the analysis from Section 4 in order to calculate the (continuous) Morava K-theory of the topological negative cyclic homology of $y(n)$. We analyze the homological homotopy fixed point spectral sequence in Section 5.1. We use the result to calculate Margolis homology and continuous Morava K-theory in Section 5.2; the main result is Theorem 5.6.

5.1. **Homological $\mathbb{T}$-homotopy fixed point spectral sequence for $THH(y(n))$.** We now analyze the homological homotopy fixed point spectral sequence converging to the continuous homology of topological negative cyclic homology of $y(n)$. This spectral sequence has the form

$$E^2(n) = H^{-*}(\mathbb{T}; H_*(THH(y(n)))) \Rightarrow H^c_*(TC^-(y(n)))$$

where

$$H^c_*(TC^-(y(n))) = \lim_{i\to\infty} H_*TC^-(y(n))[i]$$

and $TC^-(y(n))[i] := F(E\,\mathbb{T}^{(i)}_+, THH(y(n)))^{\mathbb{T}}$ so that $\lim\limits_{i\to\infty} TC^-(y(n))[i] = TC^-(y(n))$.

Most of the analysis from Section 4 carries over *mutatis mutandis*. We need the following lemma to compute the E^3-page of this spectral sequence. The key difference between the $\mathbb{T}$-Tate and $\mathbb{T}$-homotopy fixed point spectral sequences is the presence of t^{-1}, which greatly simplified $\hat{E}^3(n)$ in Section 4.4.

Proposition 5.1. *There is an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$H^c_*(TC^-(y(n))) \cong \left[P(\bar{\xi}_1^2, \ldots, \bar{\xi}_n^2) \otimes E(\bar{\xi}_2', \ldots \bar{\xi}_n') \otimes E(\bar{\xi}_n\sigma\bar{\xi}_n) \otimes P(t)\right] \oplus T$$

where T is the simple t-torsion module

$$T \cong P(\bar{\xi}_1^2, \bar{\xi}_2^2, \ldots, \bar{\xi}_n^2) \otimes P(\bar{\xi}_2', \ldots, \bar{\xi}_n') \otimes \mathbb{F}_2 \left\{ \prod_{i=1}^{n} (\sigma\bar{\xi}_i)^{\epsilon_i} \cdot (\bar{\xi}_j x_j)^{\delta} : \epsilon_k, \delta \in \{0,1\}, \epsilon_k = 0 \textit{ if } k \geq j \textit{ and } \delta = 1 \right\}$$

where $|t| = (-2, 0)$, $|\bar{\xi}_i| = (0, 2^i - 1)$, $|\sigma\bar{\xi}_i| = (0, 2^i)$, and $x_i = \sigma\bar{\xi}_i\sigma\bar{\xi}_{i+1} \cdot \ldots \cdot \sigma\bar{\xi}_n$.

Proof. The homological homotopy fixed point spectral sequence has E^2-term

$$E^2_{**}(n) = H^{-*}(\mathbb{T}; H_*(THH(y(n)))) \cong P(t) \otimes P(\bar{\xi}_1, \ldots, \bar{\xi}_n) \otimes E(\sigma\bar{\xi}_1, \ldots, \sigma\bar{\xi}_n)$$

where $|t| = (-2, 0)$, $|\bar{\xi}_i| = (0, 2^i - 1)$ and $|\sigma\bar{\xi}_i| = (0, 2^i)$. As in the Tate case, $E^2(n)$ is a module over $E^2_{*,*}(0) \cong P(t)$. Therefore $d^r(t) = 0$ for all $r \geq 1$ and all differentials are t-linear.

The d^2-differentials in the homological homotopy fixed point spectral sequence are of the form $d^2(x) = t\sigma x$ by [21, Prop. 3.2], and σ acts as a derivation by the proof of [41, Prop. 3.2], which only relies on R being an E_1-ring spectrum. We therefore obtain an additive isomorphism

$$\ker d^2 \cong P(t) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2^2, \ldots, \bar{\xi}_n^2) \otimes P(\bar{\xi}_2', \ldots, \bar{\xi}_n') \otimes T_0$$

where

$$T_0 = \mathbb{F}_2\{1, \prod_{i=1}^{n} \sigma\bar{\xi}_i^{\epsilon_i} \cdot (\bar{\xi}_j x_j)^{\delta} : \epsilon_k, \delta \in \{0,1\}, \epsilon_k = 0 \text{ for } k \geq j \text{ when } \delta = 1\}, \tag{7}$$

and $x_i = \sigma\bar{\xi}_i\sigma\bar{\xi}_{i+1}\ldots\sigma\bar{\xi}_n$ as in Proposition 3.12.We therefore just need to compute $\operatorname{im} d^2 \subset \ker d^2$ to identify $E^3_{*,*}(n)$. First, note that $t\sigma\bar{\xi}_i$ is in $\operatorname{im} d^2$ for all $1 \leq i \leq n$ by the differentials $d^2(\bar{\xi}_i) = t\sigma\bar{\xi}_i$. It is also clear that no element of the form $x \in P(t) \otimes P(\bar{\xi}_1^2, \ldots, \bar{\xi}_n^2) \otimes P(\bar{\xi}_2', ..., \bar{\xi}_n')$ is in $\operatorname{im} d^2$. We observe that $t\bar{\xi}_i x_i + t\bar{\xi}_j x_j \in \operatorname{im} d^2$ for all distinct i, j such that $1 \leq i, j \leq n$ as in the proof of Proposition 4.6. We conclude that

$$\operatorname{im} d^2 = \left[P(t) \otimes E(\sigma\bar{\xi}_1, \ldots, \sigma\bar{\xi}_n) \otimes \mathbb{F}_2\{y \cdot t\bar{\xi}_i x_i + yt\bar{\xi}_j x_j : 1 \leq i < j \leq n \text{ and } y \in \ker d^2\}\right]\{t\}.$$

Thus, up to a change of basis, $t\bar{\xi}_n x_n$ survives to $E^3_{*,*}(n)$. We can therefore identify the E^3-page as

$$E^3_{**}(n) \cong P(\bar{\xi}_1^2, \ldots, \bar{\xi}_n^2) \otimes E(\bar{\xi}_2', \ldots \bar{\xi}_n') \otimes E(\bar{\xi}_n x_n) \otimes P(t) \oplus T$$

with T as defined in the statement of the proposition.

To see that there are no further differentials, we use the map of homological $\mathbb{T}$-homotopy fixed point spectral sequences induced by the $\mathbb{T}$-equivariant map $THH(y(n)) \to THH(H\mathbb{F}_2)$. The homological homotopy fixed point spectral sequence converging to $H_*^c(TC^-(H\mathbb{F}_2))$ has E^3-page

$$E^3_{**}(\infty) \cong P(t) \otimes P(\bar{\xi}_{i+1}' : i \geq 1) \oplus P(\bar{\xi}_{i+1}' : i \geq 1) \otimes \mathbb{F}_2\{(\sigma\bar{\xi}_1)^k\}_{k \geq 1}.$$

By Proposition 3.12 the map is injective on E^3-pages and there is an isomorphism $E^3_{*,*}(\infty) \cong E^\infty_{*,*}(\infty)$ by [21, Thm. 5.1]. Consequently, there are isomorphisms $E^3_{*,*}(n) \cong E^\infty_{*,*}(n)$ for all $n > 0$. Note that we are are only claiming an isomorphism of graded $\mathbb{F}_p$-vector spaces, so there are no hidden extensions we need to consider. ☐

As in our analysis of the Tate construction, we can easily deduce the homology of the truncations of the homotopy fixed point spectra $TC^-(y(n))[i]$.

Corollary 5.2. *There is an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$H_*(TC^-(y(n))[i]) \cong [P(\bar{\xi}_1, \ldots, \bar{\xi}_n)\{t^i\}] \oplus [P(\bar{\xi}_1^2, \ldots, \bar{\xi}_n^2) \otimes E(\bar{\xi}_2', \ldots, \bar{\xi}_n') \otimes E(\bar{\xi}_n\sigma\bar{\xi}_n) \otimes P(t)/(t^{i+1})] \oplus T$$

with the same bidegrees and module T as the statement of Proposition 5.1.

5.2. **Chromatic complexity of $TC^-(y(n))$.** We now modify our analysis from Section 5.1 to study the chromatic complexity of the topological negative cyclic homology $TC^-(y(n))$.

Lemma 5.3. *There are isomorphisms of continuous E_*-comodules*

$$H_*^c(TC^-(y(n))) \cong H_*(z(n)/v_n) \otimes P(t) \oplus T$$

where T is defined as in the statement of Propositiion 5.1 and

$$H_*^c(TC^-(y(n))[i]) \cong H_*(y(n))\{t^i\} \oplus H_*(z(n)/v_n) \otimes P(t)/t^{i+1} \oplus T$$

where the simple t-torsion classes $\sigma\bar{\xi}_i$ are E_*-comodule primitives and the coaction on $\bar{\xi}_j x_j$ has the form

$$\nu_n(\bar{\xi}_j x_j) = \nu_\infty(\bar{\xi}_j x_j) + \sum_{i=1}^{n+1} \bar{\xi}_i \otimes \bar{\xi}_{n+1-i}^{2^i}.$$

Proof. A straightforward modification of the proof of Lemma 4.13 can be used to calculate the E_*-coaction on $H_*^c(TC^-(y(n)))$. The only classes which were not discussed in Lemma 4.13 are monomials of the form $\sigma\bar{\xi}_{i_1}\cdots\sigma\bar{\xi}_{i_k}$, the classes $\bar{\xi}_j x_j$ for $1 \le j \le n-1$, and nontrivial products of these elements.

The classes $\sigma\bar{\xi}_{i_1}\cdots\sigma\bar{\xi}_{i_k}$ map to the classes of the same name in $H_*^c(TC^-(H\mathbb{F}_2))$ by Proposition 3.12, so we have $\nu_n(\sigma\bar{\xi}_i) = \nu_\infty(\sigma\bar{\xi}_i)$ and we see that $\sigma\bar{\xi}_i$ is a comodule primitive by our choice of basis prior to Proposition 3.12. The classes $\bar{\xi}_j x_j$ map to $\bar{\xi}_j x_j + \bar{\xi}_{n+1}$ by Proposition 3.12, which proves the coaction stated in the lemma. □

Lemma 5.4. *The Margolis homology of $TC^-(y(n))$ is isomorphic to*

$$H(TC^-(y(n))[i]; Q_m) \begin{cases} 0 & \text{if } 1 \le m \le n-1, \\ \Sigma^{-2i} H_*(y(n)) & \text{if } m = n \\ H_*(TC^-(y(n))[i]) & \text{if } m \ge n+1. \end{cases}$$

Proof. The decomposition in Corollary 5.2 is a splitting as $E(Q_m)$-modules since t does not appear in the coaction of any class in the second summand and $Q_m(t) = 0$. Thus the Margolis homology decomposes as

$$\begin{aligned} H(TC^-(y(n))[i]; Q_m) \cong & H(H_*(y(n))\{t^i\}; Q_m) \oplus H(H_*(z(n)/v_n) \otimes P(t)/(t^i)\{t\}; Q_m) \\ & \oplus H(H_*(z(n)/v_n) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2^2, \ldots, \bar{\xi}_n^2) \otimes P(\bar{\xi}_2', \ldots, \bar{\xi}_n') \otimes T_0) \end{aligned}$$

where T_0 is defined as in Equation (7).

We define

$$\begin{aligned} M_1 &:= H_*(y(n))\{t^i\}, \\ M_2 &:= H_*(z(n)/v_n) \otimes P(t)/(t^i)\{t\}, \\ M_3 &:= H_*(z(n)/v_n) \otimes P(\bar{\xi}_1^2, \bar{\xi}_2^2, \ldots, \bar{\xi}_n^2) \otimes P(\bar{\xi}_2', \ldots, \bar{\xi}_n') \otimes T_0 \end{aligned}$$

to be the first, second, and third summands, respectively.

The Margolis homology of M_1 was computed in Lemma 2.16. The Margolis homology of M_2 can be computed the same way that $H(TP(y(n))[i]; Q_m)$ was computed in Section 5.1, giving

$$H(M_2; Q_m) \cong \begin{cases} 0 & \text{if } 1 \le m \le n, \\ P(\bar{\xi}_1^2, \bar{\xi}_2, \ldots, \bar{\xi}_n) \otimes E(\bar{\xi}_{n+1}) \otimes P(t)/(t^{i+1}) & \text{if } m = n+1. \end{cases}$$

Finally, the Margolis homology of M_3 can be computed as follows. For $1 \le m \le n$, we have $H(H_*(z(n)/v_n); Q_m) = 0$ by Corollary 2.19, so $H(M_3; Q_m) = 0$ by the Künneth formula. For $m \ge n+1$, we observe that there are no terms of the form $\bar{\xi}_m \otimes y$ in the coproduct of any classes. Indeed, these terms could only come from the coproducts of complementary σ-cycles, but the coproduct of any complementary σ-cycle z contains only terms of the form $\bar{\xi}_j^2 \otimes y$ where $j \ge 0$; compare with the proof of Lemma 4.13. □

Remark 5.5. One can also calculate $H(TC^-(y(n))[i]; Q_0)$ explicitly; however, the answer is more complicated than $H(TP(y(n)); Q_0)$ since we need to account for the t-torsion classes appearing in $H_*(TC^-(y(n))[i])$. Since we will not need it in the sequel, we omit it.

Theorem 5.6. *The continuous Morava K-theory of $TC^-(y(n), H\mathbb{F}_2)$ is given by*

$$K(m)_*^c(TC^-(y(n), H\mathbb{F}_2)) \cong 0$$

for $1 \le m \le n$.

Proof. Let $TC^-(y(n), H\mathbb{F}_2)[i]$ be the fiber in the fiber sequence

$$TC^-(y(n), H\mathbb{F}_2)[i] \to TC^-(y(n))[i] \to TC^-(H\mathbb{F}_2)[i].$$

The long exact sequence in Morava K-theory $K(m)$ associated to this fiber sequence proves the result. □

6. Topological cyclic homology and algebraic K-theory of $y(n)$

In this section, we prove the main result of the paper, Theorem 6.10, which says that the relative algebraic K-theory $K(y(n), H\mathbb{F}_2)$ has chromatic complexity at least $n+1$. In Section 6.1, we determine the chromatic complexity of the relative topological cyclic homology $TC(y(n), H\mathbb{F}_2)$, modulo a conjecture about continuous Morava K-theory, by applying the formula for topological cyclic homology given by Nikolaus and Scholze [43, Prop. II.1.9]. In Section 6.2, we apply the Dundas-Goodwillie-McCarthy Theorem [22, Thm. 2.2.1] to determine the chromatic complexity of $K(y(n), H\mathbb{F}_2)$.

Throughout this section, we assume the following conjecture which is considered in work in progress of the first author and Salch [5].

Conjecture 6.1. *Let $0 \leq n \leq \infty$ and $1 \leq m \leq n$. Then there are isomorphisms*

$$K(m)^c_*(TP(y(n))) \cong K(m)_*(TP(y(n)))$$

and

$$K(m)^c_*(TC^-(y(n))) \cong K(m)_*(TC^-(y(n))).$$

Remark 6.2. We note that we do not expect these isomorphisms to hold for the Morava K-theory of all spectra, but in fact they depend on properties of the filtered spectra $\{TP(y(n))[i]\}_{i\in\mathbb{Z}}$ and $\{TC^-(y(n))[i]\}_{i\in\mathbb{Z}}$.

Assuming Conjecture 6.1, we can apply the main results from Section 4 and Section 5 to prove the following theorem.

Theorem 6.3. *The relative topological periodic cyclic homology of $y(n) \to H\mathbb{F}_2$ and the relative topological negative cyclic homology of $y(n) \to H\mathbb{F}_2$ are $K(m)$-acyclic for $0 \leq m \leq n$; i.e. there are isomorphisms*

$$K(m)_*(TP(y(n), H\mathbb{F}_2)) \cong K(m)_*(TC^-(y(n), H\mathbb{F}_2)) \cong 0$$

for $1 \leq m \leq n$.

Proof. The result easily follows from Theorem 4.24, Theorem 5.6, and Conjecture 6.1. □

6.1. Chromatic complexity of $TC(y(n))$.

By work of Nikolaus-Scholze [43], topological cyclic homology can be computed directly from topological negative cyclic homology and topological periodic cyclic homology. We first recall their setup.

Definition 6.4. [43, Def. II.1.1] A *cyclotomic spectrum* is a spectrum X with $\mathbb{T}$-action equipped with a $\mathbb{T}$-equivariant map $\varphi_p \colon X \to X^{tC_p}$ for each $p \in \mathbb{P}$ where $\mathbb{P}$ is the set of all prime numbers.

By [43, Sec. II.6], there is an equivalence of ∞-categories between the ∞-category of genuine cyclotomic spectra [43, Sec. II.3]) and the ∞-category of cyclotomic spectra as defined above. (Here we use the term ∞-category for a quasicategory following the convention in [43].) In particular, $THH(R)$ is a cyclotomic spectrum for any E_1-ring spectrum R. Below we will write

$$TC^-(X) \xrightarrow{can} TP(X)$$

for the canonical map $X^{h\,\mathbb{T}} \to X^{t\,\mathbb{T}}$ and $\widehat{TP}(X)$ for $\prod_{p\in\mathbb{P}} \widehat{TP}(X)_p$.

Theorem 6.5. [43, Prop. II.1.9] *Let $(X, \{\varphi_p\}_{p\in\mathbb{P}})$ be a bounded below cyclotomic spectrum. There is a functorial fiber sequence*

$$TC(X) \longrightarrow TC^-(X) \xrightarrow{\prod_{p\in\mathbb{P}}(\varphi_p^{h\,\mathbb{T}}-can)} \widehat{TP}(X).$$

Implicit in this statement of the theorem is the equivalence

$$\widehat{TP}(X) \simeq \prod_{p\in\mathbb{P}}(X^{tC_p})^{h\,\mathbb{T}}$$

of [43, Rem. II.4.3], which relies on the assumption that X is bounded below.

If $A \to B$ is a map of connective E_1-ring spectra, then there is a fiber sequence

$$TC(A,B) \longrightarrow TC^-(A,B) \xrightarrow{\prod_{p\in\mathbb{P}}(\varphi_p^{h\,\mathbb{T}}-can)} \widehat{TP}(A,B).$$

This fiber sequence induces a long exact sequence in Morava K-theory $K(m)$. Also note that when A is p-complete, $K(m)_*\widehat{TP}(A) \cong K(m)_*TP(A)$ for $m \geq 1$. The following theorem is then an easy consequence of Theorem 6.3 and Proposition 6.5

Theorem 6.6. *Let $1 \leq m \leq n$. The relative topological cyclic homology $TC(y(n), H\mathbb{F}_2)$ is $K(m)$-acyclic, i.e.*

$$K(m)_*(TC(y(n), H\mathbb{F}_2)) \cong 0.$$

We emphasize that this theorem relies on the previous theorems and Conjecture 6.1. We also note that it is not possible to prove this theorem by computing the continuous Morava K-theory of $TC(y(n))$ and then applying the analog of Assumption 6.1 for topological cyclic homology. In fact, we cannot define a filtration of $TC(R)$ by taking homotopy equalizers of the skeletal filtration of $TC^-(R)$ and the Greenlees filtration of $TC(R)$ levelwise. For X a bounded below p-complete p-cyclotomic spectrum, the map $\varphi_p^{h\,\mathbb{T}}$ used to define $TC(X)$ is

$$\varphi_p^{h\,\mathbb{T}}\colon\, TC^-(X) \to (X^{tC_p})^{h\,\mathbb{T}} \xleftarrow{\simeq} TP(X),$$

by [43, Prop. II.1.9], where the last equivalence follows from [43, Rem. II.4.3] and the assumption that X is p-complete. In order to prove the last equivalence, one needs to know that the canonical morphism $X^{t\,\mathbb{T}} \to (X^{tC_p})^{h\,\mathbb{T}}$ is p-completion, proven in [43, Lem. II.4.2]. This identification fails when one restricts to truncations; in other words, this last equivalence cannot be expressed as an equivalence of filtered objects where the source is filtered using the skeletal filtration and the target is filtered using the Greenlees filtration. This is illustrated in the following example.

Example 6.7. Let $X = THH(H\mathbb{F}_2)$ with the usual $\mathbb{T}$-action. The truncated homotopical Tate spectral sequence converging to $\pi_*(X^{t\,\mathbb{T}})$ collapses for bidegree reasons, so we see that $\pi_*(X^{t\,\mathbb{T}}[i])$ is given by $P(t^{-1})\{t^i\} \otimes P(u)$, up to possible hidden multiplicative extensions, with $|t| = -2$ and $|u| = 2$. In particular, the homotopy groups of $X^{t\,\mathbb{T}}[i]$ are bounded below. On the other hand, the spectrum X^{tC_2} has homotopy

$$\pi_*(X^{tC_2}) \cong P(\hat{u}, \hat{u}^{-1})$$

with $|\hat{u}| = 2$ by [29, Eq. 4.3.4]. The truncated homotopical homotopy fixed point spectral sequence converging to $\pi_*((X^{tC_2})^{h\,\mathbb{T}}[i])$ collapses for bidegree reasons, so we can compute

$$\pi_*((X^{tC_2})^{t\,\mathbb{T}}[i]) \cong P(t)\{t^{-i}\} \otimes P(\hat{u}, \hat{u}^{-1})$$

up to hidden multiplicative extensions. The homotopy groups are not bounded below, so the truncated spectra are not equivalent (note that the we did not need to know any hidden multiplicative extensions to conclude this). In fact, this shows that there is no choice of finite truncation of $X^{t\,\mathbb{T}}$ in the Greenlees filtration which is equivalent to $(X^{tC_p})^{h\,\mathbb{T}}[i]$ after p-completion.

6.2. **Chromatic complexity of $K(y(n))$.** Using Theorem 6.6, we now compute Morava K-theory of the algebraic K-theory of $y(n)$. We begin with the rational case.

Lemma 6.8. *The relative algebraic K-theory $K(y(n), H\mathbb{F}_2)$ is rationally acyclic.*

Proof. Since $y(n) \longrightarrow H\mathbb{F}_2$ is a rational equivalence, $K(y(n)) \to K(H\mathbb{F}_2)$ is a rational equivalence by [51, Prop. 2.2] (see [12, Sec 2.3] for further discussion). □

For the remaining cases, we apply the Dundas-Goodwillie-McCarthy theorem [22, Thm 2.2.1], which we recall below.

Theorem 6.9 (Dundas-Goodwillie-McCarthy)**.** *If $A \to B$ is a map of connective E_1-ring spectra such that the map $\pi_0 A \to \pi_0 B$ is surjective and has nilpotent kernel, then there is an equivalence*

$$K(A, B) \simeq TC(A, B).$$

We can now prove that relative algebraic K-theory increases chromatic complexity for $y(n)$, again assuming Conjecture 6.1.

Theorem 6.10. *For $0 \leq m \leq n$, the m-th Morava K-theory of the relative algebraic K-theory $K(y(n), H\mathbb{F}_2)$ vanishes, i.e.*

$$K(m)_*(K(y(n), H\mathbb{F}_2)) \cong 0$$

Proof. This follows from Theorem 6.6, Lemma 6.8, and Theorem 6.9. □

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