FUNDAMENTAL THEOREMS OF ALGEBRAIC K-THEORY

GABE ANGELINI-KNOLL

1. Additivity

Algebraic K-theory has deep connections to number theory and geometry that we would like to explore. Unfortunately, it is often difficult to compute. For example, the algebraic K-theory of the integers, which one might expect to be a basic result, is still not completely known. Therefore, anything that helps reduce algebraic K-theory to something more computable is useful. We will build up to a result called "the fundamental theorem of algebraic K-theory" that uses many of the other important theorems in the proof.

Most of these theorems relate the algebraic K-theory of two similar categories where one is a little more computable than the other and the theorem tells us that either they have homotopy equivalent algebraic K-theory or at least we can understand the fiber. The first result I want to talk about deviates from that, but it can be seen as one of the most basic fundamental results, since Staffeldt [5] was able to prove many of the other fundamental theorems from this result and common tools from homotopy theory.

Definition 1.1. Let $\mathcal{C}$ be a Waldhausen category, define $\mathcal{E}(\mathcal{C})$ to be the category with objects, cofiber sequences in $\mathcal{C}$, and morphisms given levelwise. This category can be given the structure of a Waldhausen category.

Theorem 1.2 (Extension)**.**

$$K(\mathcal{E}(\mathcal{C})) \simeq K(\mathcal{C}) \vee K(\mathcal{C})$$

Proof. See McCarthy [3] or Waldhausen's original paper [7]. □

This theorem is one version of additivity, though sometimes it is referred to as the extension theorem and the following corollary is referred to as Additivity.

Corollary 1.3. *(Additivity) An exact sequence of exact functors $F" \longrightarrow F \longrightarrow F'$ from $\mathcal{C}$ to $\mathcal{D}$ induces*

$$F"_* + F'_* \simeq F_* : K(\mathcal{C}) \longrightarrow K(\mathcal{D}).$$

Proof. (A sketch) The extension theorem implies that the exact sequence $s \longrightarrow t \longrightarrow q$ of functors from $\mathcal{E}(\mathcal{D})$ to $\mathcal{D}$ satisfies $t \simeq s \coprod q$. Then use the fact that this is the universal example in the sense that giving an exact sequence of functors from $\mathcal{C}$ to $\mathcal{D}$ is the same as giving an exact functor from $\mathcal{C}$ to $\mathcal{E}(\mathcal{D})$. □

When $\mathcal{C}$ and $\mathcal{D}$ are exact categories, it is clear what we mean by an exact functor and an exact sequence of functors is a sequence that induces an exact sequence in $\mathcal{D}$ when evaluated at any object C in $\mathcal{C}$. If $\mathcal{C}$ and $\mathcal{D}$ are Waldhausen categories, then an exact functor is one that preserves cofibrations, weak equivalences, and pushouts. Similarly, an exact sequence of functors is one that produces a cofiber sequence in the target category when evaluated on any object in the source category plus an additional pushout condition.

We can view additivity as revealing the property that algebraic K-theory splits exact sequences. In fact algebraic K-theory can be seen as the universal functor that splits exact sequences. This has been made rigorous for stable infinity categories by Blumberg-Gepner-Tabuada [2], and Waldhausen infinity categories by Barwick [1]. It has been known for a long time in degree zero since K_0 is the universal target of Euler characteristics.

2. Resolution and Devissage

These next two theorems are for exact and abelian categories. For both of them we are given two exact or abelian categories where the less computable one is built out of the more computable one in some way.

Theorem 2.1 (Resolution)**.** *Suppose $\mathcal{P} \subset \mathcal{H}$ are exact categories with $\mathcal{P}$ an exact subcategory of $\mathcal{H}$ and $\mathcal{P}$ is closed under extensions, kernels of admissible surjections in $\mathcal{H}$, then if in addition for any $M \in \mathcal{H}$, there is a finite resolution,*

$$P_n \longrightarrow P_{n-1} \longrightarrow ... \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

of M by objects P_i in $\mathcal{P}$, then

$$K(\mathcal{P}) \simeq K(\mathcal{H})$$

There is one application that is particularly important. First, let $M(R)$ be the category of finitely generated R-modules and $H(R)$ be the category of finitely generated R-modules that admit a finite resolution by finitely generated projective modules. The G-theory $G(R) := K(M(R))$. If R is regular and Noetherian, then every R-module has finite projective dimension so $H(R) = M(R)$. Then by the Resolution theorem, $K(R) \simeq K(H(R))$ so $K(R) \simeq G(R)$.

Another important theorem of a similar flavor is Devissage.

Theorem 2.2 (Devissage)**.** *Suppose $\mathcal{A} \subset \mathcal{B}$ are abelian categories with $\mathcal{A}$ an exact abelian subcategory of $\mathcal{B}$. Let $\mathcal{A}$ be closed under subobjects and quotients. Then suppose for any $B \in \mathcal{B}$, there is a finite filtration,*

$$0 = B_m \subset B_{m-1} \subset ... \subset B_1 \subset B$$

with each filtration quotient $B_i/B_{i+1} \in \mathcal{A}$, then

$$K(\mathcal{A}) \simeq K(\mathcal{B})$$

An easy application of this result: if R is a Noetherian ring and I is a nilpotent ideal then $G(R) \simeq G(R/I)$.

Remark. It is still an open probelem to generalize Devissage to Waldhausen categories. There is some progress in this direction, for example Blumberg-Mandell have proved a Devissage theorem for ring spectra.

3. Localization theorems

For abelian categories, we have the following localization theorem.

Definition 3.1. A Serre subcategory of an abelian category is a sub-abelian category closed under subobjects, quotients and extensions.

Definition 3.2. Given a Serre subcategory $\mathcal{A}$ of an abelian category $\mathcal{B}$, we can form the quotient category $\mathcal{B}/\mathcal{A}$. The objects of $\mathcal{B}/\mathcal{A}$ are the same as the objects of $\mathcal{B}$, but the morphisms $B \longrightarrow C$ are given by equivalence classes of diagrams

$$B \longleftarrow A' \longrightarrow C$$

where $B \longleftarrow A'$ is a $\mathcal{A}$-isomorphism; i.e. the kernel and cokernel are in $\mathcal{A}$. We say the morphism

$$B \longleftarrow A'' \longrightarrow C$$

is in the same equivalence class as the first morphism if there is a diagram

$$\begin{tikzcd}
& A' \arrow[dl] \arrow[dr] & \\
B & A \arrow[l] \arrow[r] \arrow[u, "f"] \arrow[d, "f'"] & C \\
& A'' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where f and f' are $\mathcal{A}$-isomorphisms.

Theorem 3.3 (Localization)**.** *Let $\mathcal{A} \subset \mathcal{B}$ be a Serre subcategory of an abelian category. Then*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A})$$

is a homotopy fiber sequence.

There is a nice application of this result related to number theory. Let R be a Dedekind domain and Q its quotient field, then there is a long exact sequence,

$$K_{i+1}(Q) \longrightarrow \coprod_m K_i(R/m) \longrightarrow K_i(R) \longrightarrow K_i(Q) \longrightarrow \dots .$$

To state the fibration theorem (localization for Waldhausen categories) we need a couple definitions.

Definition 3.4. Given a Waldhausen category $\mathcal{C}$. A cylinder functor is a functor from $\mathrm{Ar}\mathcal{C}$ to small diagrams in $\mathcal{C}$, sending $f : A \longrightarrow B$ to

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr] & T(f) & B \arrow[l, "j_2"'] \arrow[dl] \\ & B \arrow[u] & \end{tikzcd}$$

It must satisfy,

(1) $T(0 \longrightarrow A) = A$
(2) The functor from $\mathrm{Ar}\mathcal{C} \longrightarrow F_1\mathcal{C}$, sending f to $j_1 \coprod j_2$ is an exact functor. (Objects in $F_1\mathcal{C}$ are cofibrations and maps are natural transformation of diagrams.)

We say it satisfies the cylinder axiom if, in addition, $p : T(f) \longrightarrow B$ is a weak equivalence in $\mathcal{C}$.

Example 3.5. *In $R_f(*)$, the Waldhausen category of finite based CW-complexes, the cylinder functor is the usual based mapping cylinder.*

Definition 3.6. We say $\mathcal{C}$ satisfies Extension if for any $f : E \longrightarrow E'$ a map of cofiber sequences if the source and quotient maps are weak equivalences than so is the total map.

$$\begin{tikzcd} E : & A \arrow[r] \arrow[d, "f_1"'] & B \arrow[r] \arrow[d, "f_2"'] & C \arrow[d, "f_2"'] \\ E' : & A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$

Definition 3.7. We say $\mathcal{C}$ is saturated if for any composable pair of morphisms f, g, then if $f \circ g$ is a weak equivalence then whenever f or g is a weak equivalence then so is the other.

Usually in a Waldhausen category we keep the cofibrations fixed, but it is often useful to consider two types of weak equivalences on the same Waldhausen category.

Theorem 3.8 (Fibration)**.** *Let $\mathcal{C}$ be a category with cofibrations and two classes of weak equivalences v and w such that $v \subset w$, $(\mathcal{C}, v)$ and $(\mathcal{C}, w)$ form Waldhausen categories. Suppose $(\mathcal{C}, w)$ has a cylinder functor satisfying the cylinder axiom and the weak equivalences satisfy saturation and extension. Then there is a fiber sequence,*

$$K(\mathcal{C}^w) \longrightarrow K(\mathcal{C}, v) \longrightarrow K(\mathcal{C}, w)$$

where $\mathcal{C}^w$ is the subcategory of $(\mathcal{C}, v)$ containing objects $C \in \mathcal{C}$ such that $0 \longrightarrow C$ is a weak equivalence in w.

Example 3.9. *Let L_E and $L_{E'}$ be two localizations of the category of finite spectra where $L_{E'}$ is coarser than L_E. Then if we let v be the the maps which induce an equivalence in*

E' and w be maps that induce an equivalence in E, we have a fiber sequence

$$K(\mathcal{C}^w) \longrightarrow K(L_E fS) \longrightarrow K(L_{E'} fS)$$

where the f indicates that we consider only those E-localizations of finite spectra that are again connected by a zigzag of weak equivalences to a finite spectrum. The fiber is the category of E'-acyclic spectra in $L_E fS$.

4. The fundamental theorem of algebraic K-theory

We would like to understand if algebraic K-theory is homotopy invariant in some sense. In particular, we'd like to know if algebraic K-theory is $\mathbb{A}^1$ invariant; i.e. if we think of algebraic K-theory of schemes then $K(X \times \mathbb{A}^1) \simeq K(X)$. It turns out that this is not true in general, but it is for a certain class of rings.

Theorem 4.1 (Fundamental Theorem of algebraic K-theory)**.** *Suppose R is a regular Noetherian ring, then*

(1) $K(R) \simeq K(R[s])$
(2) $K(R[s, s^{-1}]) \simeq K(R) \vee \Sigma K(R)$

Proof. (1) ⇒ (2) : The theorem will follow from the proof for G-theory since the functors K and G are equivalent on regular Noetherian rings by the resolution theorem. Consider the localization sequence

$$G_i(R) \longrightarrow G_i(R[s]) \longrightarrow G_i(R[s, s^{-1}]) \longrightarrow G_{i-1}(R) \longrightarrow ...$$

By the Additivity theorem, the exact sequence of functors $i^* \longrightarrow i^* \longrightarrow (s = 0)_*$ which sends M to $0 \longrightarrow M[s] \longrightarrow M[s] \longrightarrow M \longrightarrow 0$ gives $(s = 0)_* \simeq 0$, so the long exact sequence splits into short exact sequences. Then we have $G(R[s]) \simeq G(R)$ and the short exact sequnece is split by the map $(s = 1)^*$.

Proof of (1): Let $S = R[st, t] \subset R[s, t]$ where $|s| = 0$ $|t| = 1$. Define $M^b_{gr}(S)$ to be the Serre subcategory of graded t-torsion modules. Devissage implies $K(M^b_{gr}(S)) \simeq K(M_{gr}(S/tS))$. The quotient category $M_{gr}(S)/M^b_{gr}(S)$ can be identified with $M(R[s])$ via the exact functor sending M to $M/(1-t)M$. Since S and S/tS are flat over R and R has finite flat dimension over each of them, $K_i(M_{gr}(S))$ and $K_i(M_{gr}(S/tS))$ are isomorphic to $G_i(R)[\sigma, \sigma^{-1}]$. Now again by additivity we can show that the map between them is $1 - \sigma$. Thus, $G_n(R[s]) \cong G_n(R)$. □

5. Generalizations

There are two ways we might try to generalize this result.

(1) What happens when R is not regular and Noetherian?
(2) Is there a functor similar to algebraic K-theory that is $\mathbb{A}^1$-invariant and satisfies a fundamental theorem?

For the first question, we define $NK_i^{\pm}(R) := \ker(\ K_i(R[s^{\pm 1}]) \xrightarrow{(s=1)^*} K_i(R))$. Then,

Theorem 5.1. *There is a splitting*

$$K_i(R[s, s^{-1}]) \simeq K_i(R) \oplus K_{i-1}(R) \oplus NK_i^{+}(R) \vee NK_i^{-}(R).$$

The groups $NK_i^{\pm}(R)$ can be identified with $\pi_{i-1}(Nil(R))$ where $Nil(R)$ is defined to be

$$\mathrm{fib}(K(\mathbf{Nil}\ (R)) \longrightarrow K(R))$$

where **Nil**(R) is the the category of pairs (P, v) where P is a finitely generated projective module and v is a nilpotent endomorphisms of P In fact, this splitting holds for negative K-groups as well. We can define $LK(R)$ to be the cofiber of the map from the pushout $K(R[s]) \vee_{K(R)} K(R[s^{-1}])$ to $K(R[s, s^{-1}])$. We can then loop to get $\Lambda K(R) := \Omega LK(R)$. There is a natural map $K(R) \longrightarrow \Lambda K(R)$ which induces an isomorphism in homotopy above degree zero. We can then take the colimit and define

$$K^B(R) := \mathrm{colim}(K(R) \longrightarrow \Lambda K(R) \longrightarrow \Lambda^2 K(R) \longrightarrow ...).$$

A fundamental theorem holds for this functor as well,

$$K^B(R[s, s^{-1}]) \simeq K^B(R) \vee \Sigma K^B(R) \vee NK^B(R) \vee NK^B(R).$$

We might also ask for a functor which is $\mathbb{A}^1$ invariant and satisfies a fundamental theorem for all R. The functor that does this is $KH(R)$. First, we define

$$R[\Delta^\bullet] := \ R \leftleftarrows R[t_1] \Lleftarrow R[t_1, t_2] \leftleftarrows\!\leftleftarrows ...$$

where $R[t_1, t_2, ..., t_n] \cong R[t_0, t_1, ...t_n]/\Sigma t_i = 1$. We can then define,

$$KH(R) \simeq |K^B(R[\Delta^\bullet])|$$

The functor $KH(-)$ is $\mathbb{A}^1$ invariant for all rings and it satisfies the fundamental theorem for all rings. It also has the property that if R is regular and Noetherian, then $KH(R) \simeq K(R)$.

A good reference for this material is Weibel's K-book [8] or the original papers of Waldhausen [6] and Quillen [4] which have upheld the test of time well.

References

[1] C. Barwick. On the algebraic K-theory of higher categories. *ArXiv e-prints*, April 2012.
[2] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.
[3] Randy McCarthy. On fundamental theorems of algebraic K-theory. *Topology*, 32(2):325–328, 1993.
[4] Daniel Quillen. Higher algebraic K-theory: I [mr0338129]. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 413–478. Int. Press, Somerville, MA, 2010.
[5] Ross E. Staffeldt. On fundamental theorems of algebraic K-theory. *K-Theory*, 2(4):511–532, 1989.
[6] Friedhelm Waldhausen. Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 173–195. Springer, Berlin, 1984.
[7] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.
[8] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.